TRANSFORMATION OF QUASICONVEX FUNCTIONS BY SCALING: CONSTRUCTION, PROPERTIES AND USES

BY

LOAI SHAALAN

A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

APRIL 2018

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **LOAI SHAALAN** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

Dissertation Committee

Dr. Suliman Al-Homidan (Adviser)

Dr. Nicolas Hadjisavvas (Co-adviser)

Dr. Izhar Ahmad (Member)

Dr. Mohammed Alshahrani (Member)

Dr. Qamrul Hasan Ansari (Member)

Dr. Husain S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

10/5/18
Date

عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

To my family

ACKNOWLEDGMENTS

Firstly, I would like to express my sincere gratitude to everyone who contributed to this work and everyone who took part in it in a way or another. Thank you all for your support.

I am grateful for my Mother ”Um Ibrahim”, my life-coach; words cannot express how much she means to me. She is the reason I am here today, and her prayers have surrounded me all through my life.

My deepest appreciation goes to my family, my wife ”Um Obaida” and son ”Obaida” for their outstanding support, encouragement and inspiration. To all of the positive energy they provided.

A special thanks to my advisor Dr. Suliman Al-Homidan and my co-advisor Dr. Nicolas Hadjisavvas for their constant guidance, support and advice they have provided throughout my Ph.D. study and journey. I would also like to express my sincere gratitude for their patience, motivation and immense knowledge. Their valuable constructive suggestions during the planning and development of this thesis were the most fortune of my life.

I would also like to extend my thanks to My Thesis Committee members: Dr.

Izhar Ahmad, Dr. Mohammad Alshahrani and Dr. Qamrul Hasan Ansari, for their precious support, their insightful comments and encouragement all the time of research and writing of this thesis.

I would also like to thank all of my friends at KFUPM and those I have met outside campus. Thank you for making my college experience so wonderful. Words do not express my gratitude; you are all a part of something incredible.

I owe a very important debt to Mathematics and Statics Department at KFUPM who gave me access to the research facilities. Without this precious support, it would not be possible to conduct this research.

I must express my gratitude to King Fahd University of Petroleum and Minerals not only for providing the funding, which allowed me to undertake this research, but also for giving me the opportunity to be one of their students.

Finally, last but by no mean least, to everyone whom I got to know the last four years, for all of their love and encouragement all along. To everyone who made this thesis see the light.

Thanks for all of you.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

THESIS ABSTRACT

NAME:	Loai Shaalan
TITLE OF STUDY:	Transformation of Quasiconvex Functions by Scaling: Construction, Properties and Uses
MAJOR FIELD:	Mathematics
DATE OF DEGREE:	April 2018

In this thesis, we write any lower semicontinuous quasiconvex function as a composition of two functions, one of which is nondecreasing, and the other one is quasiconvex with the property that it has has no local minimum except the global minimum. In addition, the new quasiconvex function shares the set of minimizer with the original function. We also approximate any lower semicontinuous quasiconvex function by a uniformly convergent sequence of quasiconvex functions, which have the property that every local minimum is global minimum. In both of the previous two objectives, we use the notion of "adjusted sublevel set". For this reason we study the properties of the class of sublevel sets and of the sublevel set operator.

ملخص الرسالة

الاسم الكامل: لؤي محمد عبدالرحمن شعلان

عنوان الرسالة: تحويل الدوال شبه المحدبة بالتوازن والبناء والخواص والاستخدامات

التخصص: الرياضيات

تاريخ الدرجة العلمية: نيسان 2018

في هذه الرسالة، نقوم بكتابة كل دالة شبه محدبة وشبه متصلة كتركيب دالتين، أحداهما دالة غير متناقصة، والدالة الأخرى شبه محدبة، مع خاصية أن الدالة ليس لها قيمة صغرى محلية، وإنما فقط قيمة صغرى مطلقة. بالإضافة إلى ذلك فإن الدالة شبه المحدبة تشترك مع الدالة الأصلية بمجموعة النقاط التي تحقق القيمة الصغرى. كذلك نقرب الدالة شبه المتصلة وشبه المحدبة بمتتالية منتظمة التقارب من الدوال شبه المحدبة والتي تحتوي فقط على قيم صغرى مطلقة، في كلا الهدفين السابقين نستخدم مفهوم " مجموعة المستويات الفرعية المعدلة" ولهذا السبب نحن ندرس خصائص مجموعة عمليات المستويات الفرعية المعدلة.

CHAPTER 1

INTRODUCTION

The importance of quasiconvex functions arises from their applications in many fields, such as in mathematical optimization, economics, and game theory. Von Neumann was probably the first to use quasiconvexity back in 1928, as one of the conditions of the famous minimax theorem. However, the notion of quasiconvex functions was formally introduced by De Finetti in 1949, for use in mathematical economics. Since then, it is a standard assumption in optimization and mathematical economics.

1.1 Description of the Problem

One of the useful properties of the convex functions, that every local minimum is a global minimum. This property does not applicable for the quasiconvex functions. In other words, we may have $\nabla f\left(x_0\right) = 0$ and x_0 is not a global minimum of the quasiconvex function f.

Our objective in this work is to avoid the flat parts in the quasiconvex function

f. So we have to replace the function f which has flat parts in it's graph by a quasiconvex function g, which has no flat parts in it's graph.

In this thesis, we have to write any lower semicontinuous quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ as a composition

$$f = h \circ g,$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a quasiconvex function, which has no flat parts of dimension n in it's graph, and $h : g(\mathbb{R}^n) \longrightarrow \mathbb{R}\{+\infty\}$ is a nondecreasing function.

In addition to that, we will give an approximation for any lower semicontinuous quasiconvex function $f : \Omega \longrightarrow \mathbb{R}\{+\infty\}$, where $\Omega \subseteq \mathbb{R}^n$ is convex. Our approximation is a uniformly convergent sequence of neatly quasiconvex functions (i.e function that has no flat parts in it's graph).

1.2 Methodology

First of all, we need to recall the notion of the adjusted sublevel set, that was introduced for the first time by Aussel and Hadjisavvas in [1]. We will prove the lower semicontinuity of the adjusted sublevel set as a multivalued map ($S_f^a : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$), in the case of a quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, that has a closed sublevel stes $S_f(x)$ for all $x \in \mathbb{R}^n$. In addition to that, we will prove that the normal operator to the adjusted sublevel set is cone upper semicontinuous.

To achieve our objective, that we describe in Section 1.1, we will modify the definition of g-pseudoconvex function, that was introduced by Crouzeix, Eberhard and Ralph in [6]. So, we are looking for a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that each sublevel set and the corresponding strict sublevel set of g have the same closure.

We will see that the adjusted sublevel set operator satisfies a nice continuity property (it is lower semicontinuous). In addition to that, we will show in Section 4.2 that the class of all adjusted sublevel sets is totally ordered under the inclusion of sets. These properties of the adjusted sublevel set operator of our function f, will help us to construct the function g, in the composition

$$f = h \circ g.$$

The idea is to look for a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that the sublevel set of the function g, at any point $x \in \mathbb{R}^n$, is equal to the adjusted sublevel set of the function f at the same point.

1.3 The Thesis Parts

This thesis consists of following chapters:

Chapter 1 provides a short description of the problem, the methodology, and the results.

In Chapter 2 we provide three sections. In the first section we give the basic definitions and notation that we will use in the thesis, we recall the notion of multivalued map and its continuity types. Also, we recall some notions of generalized monotonicity. In the second section we discuss some types of generalized convexity, we show the relation between generalized convexity types and we recall some of their properties. Finally in the last section we review the literature that is related to our work.

In Chapter 3 we recall the definition of the adjusted sublevel sets. Then we study their properties, especially the continuity properties. We show that the adjusted sublevel set operator is lower semicontinuous if our function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is quasiconvex with closed sublevel sets, and we prove the same property for the strict sublevel set operator with weaker assumptions. Also, we will study the continuity properties of their normal cone: we show that the adjusted normal cone operator is cone upper semicontinuous and closed.

In Chapter 4 we will introduce the notion of neatly quasiconvex function (i.e a quasiconvex function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ which has no flat part of dimension n in its graph). By using the notion of the adjusted sublevel sets and their properties that we will study in Chapter 3, we will prove that every lower semicontinuous quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ can be written as a composition $h \circ g$, where $h : \operatorname{Im}(g) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is nondecreasing, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a neatly quasiconvex function. Furthermore, the class of the adjusted sublevel sets of the function f, is equal to the class of the sublevel sets of the function g. Then we will study some properties of neatly quasiconvex functions.

In Chapter 5 we will approximate any lower semicontinuous quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, by a uniformly convergent sequence of neatly quasiconvex functions $\{g_k\}_{k\in\mathbb{N}}$.

In the last chapter we will summarize the results that we have in the thesis.

CHAPTER 2

PRELIMINARIES

In this chapter we will recall and introduce the notation and the definitions that we need in our work. Then we give a background for generalized convexity. Finally we review the literature that is related to our work.

2.1 Notation and Some Background

For any $x, y \in \mathbb{R}^n$, we set $]x, y[= \{tx + (1-t)y : 0 < t < 1\}$, and $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$. Also, if $\varepsilon > 0$, we denote by $B(x, \varepsilon)$ the open ball $\{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$, and by $\overline{B}(x, \varepsilon)$ the closed ball $\{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\}$.

Given a nonempty set $A \subseteq \mathbb{R}^n$, the interior, the closure, the boundary, and the complement of A will be denoted by $\text{int}A$, $\bar{A}$, ∂A, and A^c, respectively. The convex hull, the affine hull, and the conic hull generated by the set A are denoted by $\text{conv}(A)$, $\text{aff}\,(A)$, and $\text{cone}(A)$, respectively; where

$$\text{conv}\,(A) = \{\lambda x + (1-\lambda)\,y : x, y \in A \text{ and } \lambda \in [0,1]\}\,,$$

$$\operatorname{aff}(A) = \{\lambda x + (1-\lambda)\, y : x, y \in A \text{ and } \lambda \in \mathbb{R}\},$$

and

$$\operatorname{cone}(A) = \{\lambda x : x \in A \text{ and } \lambda > 0\}.$$

Given a nonempty set $A \subseteq \mathbb{R}^n$, its support function is the function $\sigma_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_A(y) = \sup_{x \in A} \langle x, y \rangle$$

whereas its barrier cone is the cone

$$\mathrm{b}(A) = \{y \in \mathbb{R}^n : \sigma_A(y) < +\infty\}.$$

We use $\dim(A)$ to denote the dimension of $\operatorname{aff}(A)$. Also, we set

$$B(A, \varepsilon) = \{y \in \mathbb{R}^n : \operatorname{dist}(y, A) < \varepsilon\},$$

and

$$\overline{B}(A, \varepsilon) = \{y \in \mathbb{R}^n : \operatorname{dist}(y, A) \leq \varepsilon\},$$

where

$$\operatorname{dist}(x, A) = \inf\{\|x - a\| : a \in A\} \text{ for any } x \in \mathbb{R}^n.$$

The (negative) polar cone of the set A, will be denoted by A°, where

$$A^\circ = \{y \in \mathbb{R}^n : \langle y, z \rangle \leq 0 \text{ for all } z \in A\}.$$

We recall:

Definition 2.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set. A point $x \in K$ is said to be a relative interior point if there exists $\varepsilon > 0$, such that*

$$B(x, \varepsilon) \cap \operatorname{aff} K \subseteq K.$$

The set of all relative interior points is denoted by $\operatorname{ri} K$.

We can see the difference between the interior and the relative interior in the following example.

Example 2.1 *Let $A = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$. Then the relative interior of the set A is given by*

$$\operatorname{ri}(A) = \left\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 < 1\right\}.$$

But the interior of the set A is the empty set $\emptyset$.

In other words, the relative interior of a set $K \subseteq \mathbb{R}^n$ is the interior of K with respect to the relative topology induced on $\operatorname{aff} K$.

Theorem 2.2 *([18], Theorem 6.2) Let $K \subseteq \mathbb{R}^n$ be nonempty and convex, then* $\operatorname{ri} K \neq \emptyset$.

From the last theorem, we can conclude the following corollary.

Corollary 2.3 *Let $K \subseteq \mathbb{R}^n$ be nonempty and convex, then the interior of K is nonempty with respect to* aff$K$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$. The domain of f is the set dom(f)$= \{x \in \mathbb{R}^n : f(x) < +\infty\}$. We define for any $\lambda \in \mathbb{R} \cup \{+\infty\}$ the sublevel set and the strict sublevel set by, respectively,

$$S_\lambda^f = \{y \in \mathbb{R}^n : f(y) \leq \lambda\},$$

and

$$S_\lambda^{f,<} = \{y \in \mathbb{R}^n : f(y) < \lambda\}.$$

Note that we do not define these notions only for $\lambda \in \mathbb{R}$, as other authors do. Given $x \in \mathbb{R}^n$ we set for simplicity

$$S_f(x) = S_{f(x)}^f$$

and

$$S_f^<(x) = S_{f(x)}^{f,<}.$$

Also we set

$$\rho_x^f = \text{dist}(x, S_f^<(x)).$$

The epigraph of the function f will be denoted by epi$f$, and it is defined by

$$\text{epi}f = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \geq f(x)\}.$$

Definition 2.2 *A function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ *is called lower semicontinuous if* S_λ^f *is closed for all* $\lambda \in \mathbb{R}$.

The following theorem provides equivalent conditions for the lower semicontinuity.

Theorem 2.4 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ *. The following are equivalent:*

- f *is lower semicontinuous.*
- $\mathrm{epi} f$ *is closed.*
- *For all* $x \in \mathbb{R}^n$, *if* $\{x_k\}$ *converges to* x, *then* $\liminf\limits_{k\longrightarrow+\infty} f(x_k) \geq f(x)$.

See ([15], Proposition 4.4, Proposition 4.5, and Proposition 4.6).

In this thesis we will deal mainly with multivalued maps (or: set-valued maps), that is, maps which to every point $x \in \mathbb{R}^n$ associate a (possibly empty) subset $T(x) \subseteq \mathbb{R}^m$. For a multivalued map T we use the notation $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.

Given a multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we recall that the domain of T is the set

$$\mathrm{D}(T) = \{x \in \mathbb{R}^n : T(x) \neq \emptyset\}$$

and the graph of T is the set

$$\mathrm{gr}(T) = \{(x, x') \in \mathbb{R}^n \times \mathbb{R}^m : x' \in T(x)\}.$$

Definition 2.3 *A multivalued map* $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *is said to be closed, if* $\mathrm{gr}(T)$ *is closed in* $\mathbb{R}^n \times \mathbb{R}^m$. *We say that* T *is closed at* $x \in \mathbb{R}^n$, *if the following implication* *holds:*

$$\{(x_k, y_k)\} \subseteq \mathbb{R}^n \times \mathbb{R}^m \textit{ and converges to } (x, y) \implies y \in T(x).$$

Now, we recall the definitions of the continuity types for multivalued maps.

Definition 2.4 *A multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous (u.s.c.) at $x \in \mathrm{D}(T)$, if for every open set $V \subseteq \mathbb{R}^n$ such that $T(x) \subseteq V$, there exists an open set $U \subseteq \mathbb{R}^n$, such that $x \in U$ and $T(u) \subseteq V$ for all $u \in U$. A multivalued map T is said to be u.s.c. if it is u.s.c. at every $x \in \mathrm{D}(T)$.*

Theorem 2.5 *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multivalued map, then the following are equivalent:*

i) *T is u.s.c.*

ii) *For all $x \in \mathrm{D}(T)$ and every open set $V \subseteq \mathbb{R}^n$ such that $T(x) \subseteq V$, if $\{x_k\} \subseteq \mathrm{D}(T)$ converges to x, there exists $K \in \mathbb{N}$ such that $T(x_k) \subseteq V$, for all $k > K$.*

See ([11], Proposition 2.5).

Definition 2.5 *A multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be lower semicontinuous (l.s.c.) at $x \in \mathrm{D}(T)$, if for every open set $V \subseteq \mathbb{R}^n$ such that $T(x) \cap V \neq \emptyset$, there exists an open set $U \subseteq \mathbb{R}^n$, such that $x \in U$ and $T(u) \cap V \neq \emptyset$, for all $u \in U$. A multivalued map T is said to be l.s.c. if it is l.s.c. at every $x \in \mathrm{D}(T)$.*

Theorem 2.6 *If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a multivalued map, then the following are equivalent:*

i) *T is l.s.c.*

ii) *For all $x \in \mathrm{D}(T)$ and every open set $V \subseteq \mathbb{R}^m$ such that $T(x) \cap V \neq \emptyset$, if $\{x_k\} \subseteq \mathrm{D}(T)$ converges to x, there exists $K \in \mathbb{N}$ such that $T(x_k) \cap V \neq \emptyset$ for all $k \geq K$.*

iii) *If $\{x_k\} \subseteq \mathbb{R}^n$ converges to x and $y \in T(x)$, then for each $k \in \mathbb{N}$ we can find $y_k \in T(x_k)$ such that $y_k \to y$.*

See ([11], Chapter I, Proposition 2.6).

Definition 2.6 *A multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be continuous if it is both l.s.c. and u.s.c.*

Example 2.7 **i)** *Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be a multivalued map that is defined by*

$$T(x) = \begin{cases} [-|x|, |x|], & x \neq 5, \\ [-8, 10], & x = 5. \end{cases}$$

Then T is u.s.c. at $x = 5$, but it is not lower semicontinuous.

ii) *Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a multivalued map that is defined by*

$$F(x) = \begin{cases} [-|x|, |x|], & x \neq 5, \\ \{4\}, & x = 5. \end{cases}$$

Then F is l.s.c., but it is not u.s.c. at $x = 5$.

iii) *Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a multivalued map that is defined by*

$$F(x) = [0, |x|]\,, \text{ for all } x \in \mathbb{R}.$$

Then F is both u.s.c. and l.s.c., that means it is continuous on $\mathbb{R}$.

The following graphs of multivalued maps on $\mathbb{R}$ will show the kinds of continuity.

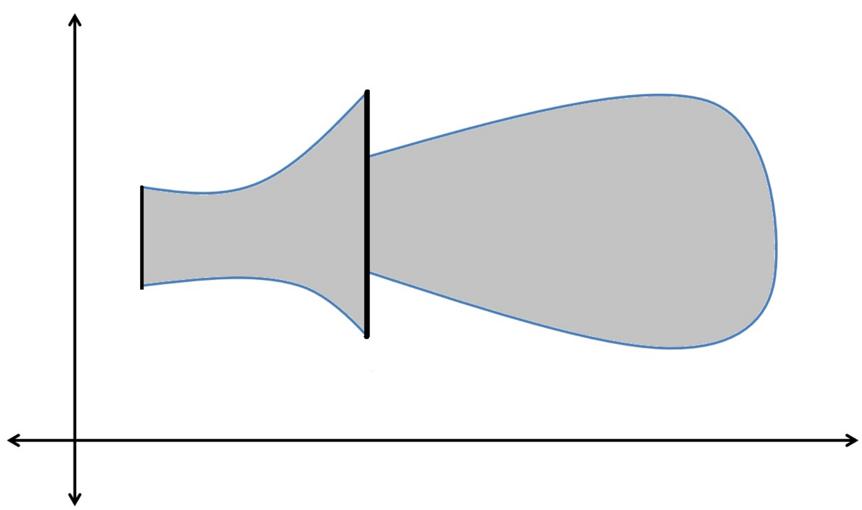

Figure 2.1: u.s.c. non l.s.c. map

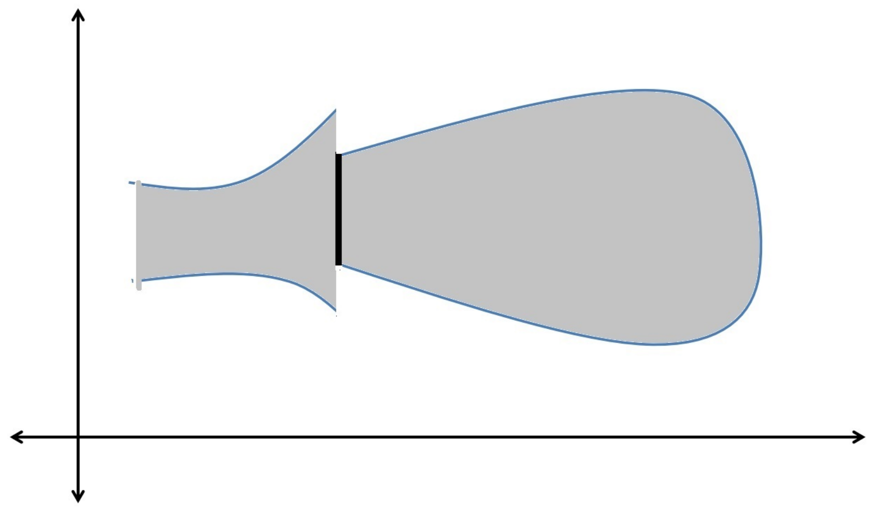

Figure 2.2: l.s.c. non u.s.c. map

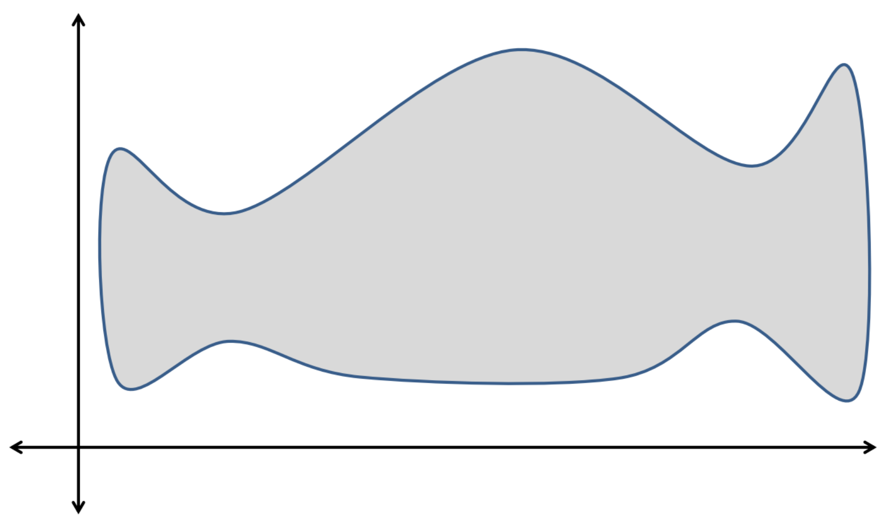

Figure 2.3: Continuous map

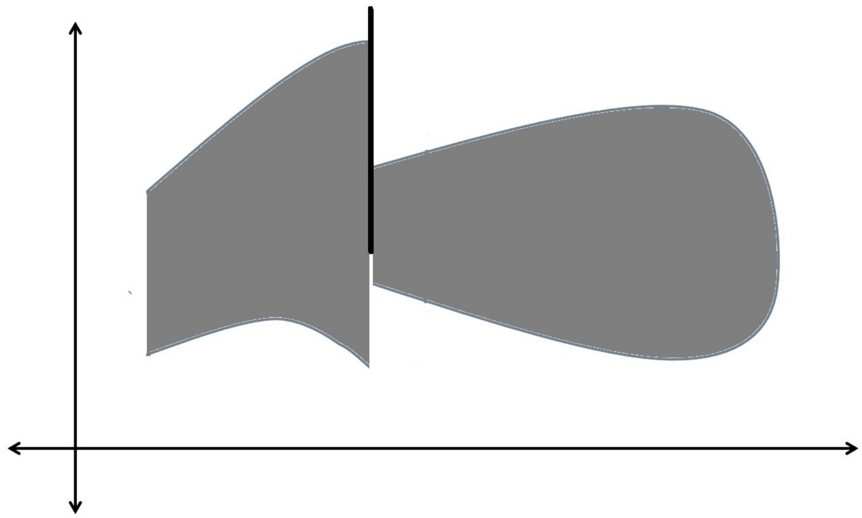

Figure 2.4: A map which is neither u.s.c. nor l.s.c.

Definition 2.7 *Let $K \subseteq \mathbb{R}^n$ and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Then we say*

i) *T is monotone on K, if for all $x, y \in K$ and $x' \in T(x)$, $y' \in T(y)$, we have*

$$\langle y - x, y' - x' \rangle \geq 0. \tag{2.1}$$

ii) *T is strictly monotone on K, if (2.1) is satisfied with strict inequality whenever $x \neq y$.*

iii) *T is pseudomonotone on K, if for all $x, y \in K$ and $x' \in T(x)$, $y' \in T(y)$, the following implication holds*

$$\langle y - x, x' \rangle \geq 0 \Longrightarrow \langle y - x, y' \rangle \geq 0$$

or equivalently

$$\langle y - x, x' \rangle > 0 \Longrightarrow \langle y - x, y' \rangle > 0.$$

iv) *T is quasimonotone on K, if for all $x, y \in K$ and $x' \in T(x)$, $y' \in T(y)$, the following implication holds*

$$\langle y - x, x' \rangle > 0 \Longrightarrow \langle y - x, y' \rangle \geq 0.$$

We can see from the last definitions that, if T is a monotone operator, then it is pseudomonotone. Also, if T is pseudomonotone, then it is quasimonotone.

For the relation between generalized monotonicity and generalized convexity, see [12] and [10].

2.2 Generalized Convexity

In this section we recall some notions of convexity and generalized convexity, and the relations between them. See [17], [15], and [3].

Definition 2.8 *Let $K \subseteq \mathbb{R}^n$ be nonempty and convex. A function $f : K \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex if*

$$f(\tau x + (1 - \tau) y) \leq \tau f(x) + (1 - \tau) f(y) \text{ for all } x, y \in K, \text{ and } \tau \in (0, 1).$$

We say that $f : K \longrightarrow \mathbb{R}$ *is strictly convex if*

$$f(\tau x + (1-\tau) y) < \tau f(x) + (1-\tau) f(y) \text{ for all } x, y \in K \text{ such that } x \neq y, \text{ and } \tau \in (0,1).$$

Let K be nonempty and convex, and $f : K \longrightarrow \mathbb{R}$. We define the extension $\widetilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ of the function f by

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in K, \\ +\infty, & x \notin K. \end{cases}$$

Proposition 2.1 *Let* $K \subseteq \mathbb{R}^n$ *be nonempty and convex. A function* $f : K \longrightarrow \mathbb{R}$ *is convex, if and only if* $\widetilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is convex.*

Proposition 2.2 *([15],* Theorem 1.37*) A function* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is convex if and only if* $\operatorname{epi} f$ *is a convex set in* $\mathbb{R}^{n+1}$.

Definition 2.9 *Let* $K \subseteq \mathbb{R}^n$ *be nonempty, open, and convex. A differentiable function* $f : K \longrightarrow \mathbb{R}$ *is pseudoconvex if for every* $x, y \in K$, *the following implication holds*

$$\langle \nabla f(x), y - x \rangle \geq 0 \text{ implies } f(y) \geq f(x).$$

The function f *is said to be strictly pseudoconvex, if for every* $x, y \in K$, *with* $x \neq y$, *the following implication holds*

$$\langle \nabla f(x), y - x \rangle \geq 0 \text{ implies } f(y) > f(x).$$

Definition 2.10 *Let $K \subseteq \mathbb{R}^n$ be nonempty and convex. A function $f : K \longrightarrow \mathbb{R}$ is strictly quasiconvex, if for all $x, y \in K$, with $x \neq y$, we have*

$$f(\tau x + (1-\tau) y) < \max\{f(x), f(y)\}.$$

Definition 2.11 *Let $K \subseteq \mathbb{R}^n$ be nonempty and convex. A function $f : K \longrightarrow \mathbb{R}$ is semistrictly quasiconvex, if for all $x, y \in K$, the following implication holds*

$$f(x) < f(y) \text{ implies } f(z) < f(y), \text{ for all } z \in]x, y[.$$

Definition 2.12 *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is quasiconvex, if for all $x, y \in \mathbb{R}^n$, we have*

$$f(\tau x + (1-\tau) y) \leq \max\{f(x), f(y)\} \text{ for all } \tau \in (0,1).$$

The following proposition provides a necessary and sufficient condition for quasiconvexity, that is probably even more useful than Definition 2.12.

Proposition 2.3 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The following are equivalent:*

a) f is quasiconvex;

b) For all $\lambda \in \mathbb{R}$, S_λ^f is convex;

c) For all $x \in \mathrm{dom}(f)$, $S_f(x)$ is convex.

The following diagram shows the relation between generalized convexity types.

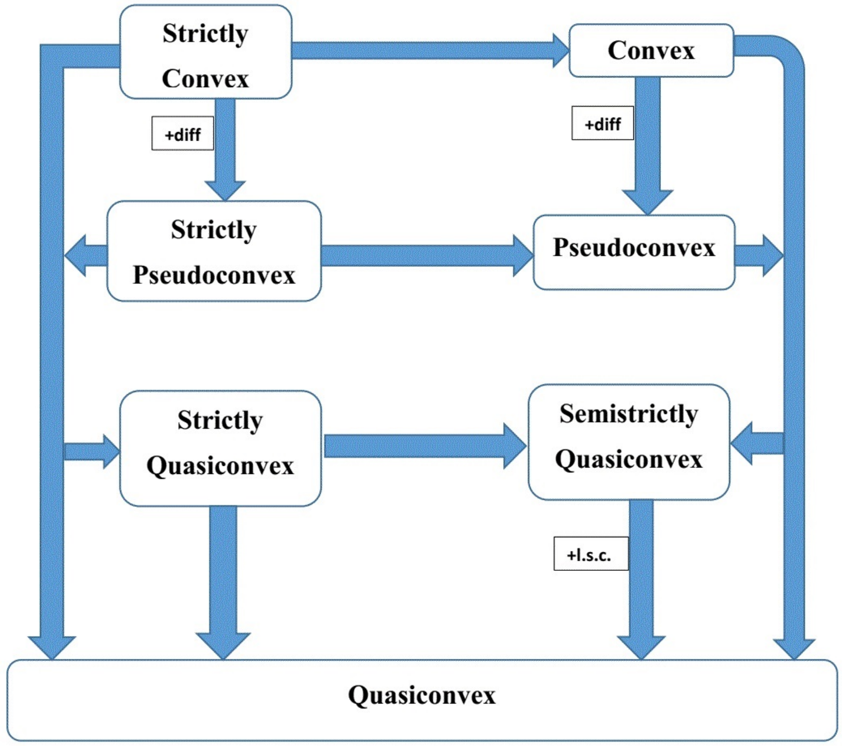

Figure 2.5: Implications between generalized convexity notions

Example 2.8 **i)** *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the function* $f(x) = x^{2n}$*. Then* f *is strictly convex for all* $n \in \mathbb{N}$*.*

ii) *The function* $f : \mathbb{R} \longrightarrow \mathbb{R}$*, defined by* $f(x) = x^3$*, is a quasiconvex function on* $\mathbb{R}$*, but is not convex.*

iii) *Take the function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *that is defined by*

$$f(x) = \begin{cases} 0, & x \in \mathbb{R}^n \setminus \{0\}, \\ 5, & x = 0. \end{cases}$$

Then f is a semistrictly quasiconvex function, but it is not quasiconvex. Note that f is not lower semicontinuous, and compare with the corresponding implication in the above diagram.

Definition 2.13 *Let $\Omega \subseteq \mathbb{R}^n$ be nonempty and convex. A function $f : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is said to be concave, quasiconcave, or pseudoconcave, if $-f$ is convex, quasiconvex, or pseudoconvex, respectively.*

2.3 Literature Review

In this section we will present some results related to our work, in chronological order.

2.3.1 Von Neumann

Most likely von Neuman was the first to use quasiconvex functions, in his well-known theorem (Minmax Theorem) in 1928, in the framwork of game theory. However, he used quasiconvexity, without choosing a name for it [19].

2.3.2 De Finetti

De Finetti was the first who used the name quasiconcave. It appeared in his article "Annali di Mathematica Pura e Applicata (1949)." The main problems that he studied were:

i) Given a totally ordered family of convex sets, is it possible to associate a convex function to this family?

ii) If f is a quasiconvex function, does there exist a monotone transformation G, such that $G(f)$ is convex?

The answer to the first problem was negative in general. But he gave a positive answer to the second problem if we assume that the first and the second derivatives are bounded. See [9].

2.3.3 Fenchel

Fenchel gave necessary and sufficient conditions for a family of convex sets, to be the family of sublevel sets of a convex function. See ([8], Chapter 3 - Section 7). The conditions are summarized in the following proposition.

Proposition 2.4 *Let* $K \subseteq \mathbb{R}^n$ *be nonempty, convex, and open, and let* $\{L_\lambda \subseteq K : \lambda \in I\}$ *be a class of subsets of* K*, where* $I \subseteq \mathbb{R}$ *is an interval. The following conditions are necessary and sufficient for the existence of a convex function* $g : K \longrightarrow \mathbb{R}$*, such that the sublevel sets of* g *are* L_λ*,* $\lambda \in I$*.*

i) $\bigcup\limits_{\lambda \in I} L_\lambda = K.$

ii) $L_{\lambda_1} \subseteq L_{\lambda_2}$ *if* $\lambda_1 < \lambda_2$*.*

iii) $\bigcap\limits_{\lambda > \lambda_0} L_\lambda = L_{\lambda_0}$*, and* $\bigcap\limits_{\lambda \in I} L_\lambda = \emptyset$ *if* I *is open to the left.*

iv) L_λ *is closed for all* $\lambda \in I$*.*

v) L_λ *is convex for all* $\lambda \in I$*.*

vi) *All the sets* L_λ*,* $\inf I < \lambda < \sup I$ *have the same barrier cone* B*.*

vii) *There is a strictly increasing continuous function* $F : I \longrightarrow \mathbb{R}$ *such that.*

$$F(\lambda_3) - F(\lambda_2) \geq [F(\lambda_2) - F(\lambda_1)]\, \chi(\lambda_1, \lambda_2, \lambda_3) \text{ for any } \lambda_1 < \lambda_2 < \lambda_3$$

where

$$\chi(\lambda_1, \lambda_2, \lambda_3) = \sup_{y \in B} \frac{\sigma(\lambda_3, y) - \sigma(\lambda_2, y)}{\sigma(\lambda_2, y) - \sigma(\lambda_1, y)},$$

and

$$\sigma(\lambda, y) = \sigma_{S_\lambda^f}(y). \ (\sigma_{S_\lambda^f} \text{ is the support function of } S_\lambda^f)$$

One of the difficulties in the last proposition, is to check the seventh condition.

2.3.4 Mereau and Paquet

Mereau and Paquet in their paper [14] gave a necessary condition, and a sufficient condition, for a twice continuously differentiable function to be pseudoconvex.

Theorem 2.9 *([14], Theorem 1) Let* $K \subseteq \mathbb{R}^n$ *be nonempty and convex. If* $f : K \longrightarrow \mathbb{R}$ *is pseudoconvex, then for all* $x \in K$ *there exists* $\alpha_x \geq 0$ *such that*

$$\nabla^2 f(x) + \alpha_x \nabla f(x) \nabla f(x)^T \geq 0.$$

Theorem 2.10 *([14], Theorem 2) Let* $K \subseteq \mathbb{R}^n$ *be nonempty and convex. And* $f : K \longrightarrow \mathbb{R}$ *is twice continuously differentiable on* K*. If there exists* $\alpha \geq 0$ *such that*

$$\nabla^2 f(x) + \alpha \nabla f(x) \nabla f(x)^T \geq 0 \text{ for all } x \in K,$$

then f is pseudoconvex on K.

2.3.5 Borde and Crouzeix

In their very influential paper [4], Borde and Crouzeix studied the sublevel sets and the normal cones on the sublevel sets of quasiconvex functions, and their continuity properties.

We will recall in this part some results from [4], that are related to our research.

Theorem 2.11 *([4], Proposition 2.1) Let $\Omega \subseteq \mathbb{R}^n$ be open and convex. If f is lower semicontinuous at $x \in \Omega$, then $N_f^<$ is closed at x.*

Theorem 2.12 *([4], Proposition 2.2) Let $\Omega \subseteq \mathbb{R}^n$ be open and convex and $f : \Omega \longrightarrow \mathbb{R}$ be lower semicontinuous at $\bar{x} \in \Omega$. Assume that there exists $\lambda < f(\bar{x})$ such that $\mathrm{int} S_\lambda^f \neq \emptyset$. Then $N_f^<$ is cone-uper semicontinuous at $\bar{x}$.*

We postpone the discussion of cone-upper semicontinuity until the next chapter.

2.3.6 Aussel and Hadjisavvas

In [1], Aussel and Hadjisavvas introduced a new kind of sublevel set. They called it "adjusted sublevel set." The adjusted sublevel set lies between the sublevel set and the strict sublevel set. The importance of the adjusted sublevel sets comes from their nice continuity properties that we will discuss in the next chapter. Also, the normal cone operator to the adjusted sublevel sets has nice properties.

It was proved in [1] that for any function f, the normal operator to the adjusted sublevel sets of f is quasimonotone, and under some conditions it may be cone upper-semicontinuous.

The definition of the adjusted sublevel set is the following.

Definition 2.14 *([1], Definition 2.3) Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ be any function. The adjusted sublevel set of f at $x\in \operatorname{dom} f\backslash \arg\min f$ is defined by*

$$S_f^a(x) = S_f(x) \cap \bar{B}\left(S_f^<(x), \rho_x^f\right)$$

where

$$\rho_x^f = \operatorname{dist}\left(x, S_f^<(x)\right).$$

The adjusted sublevel set of f at $x \in \arg\min f$ is defined by

$$S_f^a(x) = S_f(x).$$

As we could define the quasiconvexity of a function by the convexity of its sublevel sets, or by the convexity of its strict sublevel sets, we can also check the quasiconvexity of a function through the convexity of its adjusted sublevel sets.

Theorem 2.13 *([1], Proposition 2.4) Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$. Then f is quasiconvex if and only if*

$$S_f^a(x) \text{ is convex for all } x\in \operatorname{dom} f.$$

Most of the results in [1] concerned the normal cone to the adjusted sublevel stes, which is sometimes called adjusted normal cone.

Definition 2.15 *([1], Definition 2.5) Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$. *Given* $x \in \operatorname{dom} f$, *we define the normal cone to the adjusted sublevel set* $S_f^a(x)$ *at* x *by*

$$N_f^a(x) = \left\{ z \in \mathbb{R}^n : \langle z, y - x \rangle \leq 0 \textit{ for all } y \in S_f^a(x) \right\}.$$

It was shown in [1] that the operator $x \to N_f^a(x)$ has some monotonicity and continuity properties. First we recall:

Definition 2.16 *A multivalued map* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is said to be cyclically quasimonotone, if for all* $\left(x_i, x_i^{'}\right) \in \operatorname{gr} F$, $i = 1, 2, \ldots, n$, *the following implication holds:*

$$\left\langle x_i^{'}, x_{i+1} - x_i \right\rangle > 0 \textit{ for all } i = 1, 2, \ldots, n-1,$$

implies

$$\left\langle x_n^{'}, x_{n+1} - x_n \right\rangle \leq 0, \textit{ where } x_{n+1} = x_1.$$

Theorem 2.14 *([1], Proposition 3.3) For any function* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, N_f^a *is cyclically quasimonotone.*

Theorem 2.15 *([1], Proposition 3.5) Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a quasiconvex function such that*

$$\operatorname{int}\left(S_\lambda^f\right) \neq \emptyset \textit{ for all } \lambda > \inf f.$$

If f is l.s.c. at $x \in \mathrm{dom} f \backslash \arg\min f$, then N_f^a is cone upper-semicontinuous at x.

2.3.7 Crouzeix, Eberhard, and Ralph

In [6] Crouzeix, Eberhard and Ralph defined the notion of g-pseudoconvex function, which is very close to the notion of neatly quasiconvex function, that we will introduce later in this thesis. A nice property of g-pseudoconvex functions is that every local minimum is global minimum, which means that there is no flat part in the graph of the function. The letter (g) in g-pseudoconvex is an abbreviation of "geometrically".

Definition 2.17 *(See [6]) A quasiconvex function $f : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$ is said to be g-pseudoconvex, if for all $x \in \mathbb{R}^n \backslash \arg\min f$ we have*

$$\emptyset \neq \mathrm{int}\left(S_f\left(x\right)\right) \subseteq S_f^<\left(x\right) \ \text{ and } S_f\left(x\right) \subseteq \overline{S_f^<\left(x\right)}$$

From the definition of g-pseudoconvex function we can see that, if f is g-pseudoconvex, then

$$\mathrm{int}\left(S_f\left(x\right)\right) = \mathrm{int}\left(S_f^<\left(x\right)\right) \text{ and } \overline{S_f\left(x\right)} = \overline{S_f^<\left(x\right)} \text{ for all } x \in \mathbb{R}^n \backslash \arg\min f.$$

The following example shows that a g-pseudoconvex function may be not continuous.

Example 2.16 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as*

$$f(x) = \begin{cases} 3x - 4, & x < 0, \\ 0, & x = 0, \\ 2x + 5, & x > 0. \end{cases}$$

Then f is g-pseudoconvex, but it is not continuous at $x = 0$.

2.3.8 Aussel and Pistek

In their paper [2], Aussel and Pistek defined the limiting sublevel sets and their normal operator for quasiconvex functions. Then they studied the continuity properties of these operators.

Two of their results are very close to some of our results in the next chapter. In [2, Cor 1] they proved that the normal operator of a l.s.c. operator is u.s.c.. Also, in [2, Lemma 1] they proved the lower semicontinuity of the closed strict sublevel set operator ($\bar{S}^{<}(\cdot)$) in the case of l.s.c. functions.

2.3.9 Connell and Rasmusen

Connell and Rasmusen in [5] gave conditions for continuous function, to be transformable to strictly convex one, through a monotone transformation.

2.3.10 Lucchetti and Milasi

In [16], Lucchetti and Milasi approximated a quasiconvex function by a sequence of strictly quasiconvex functions. Their theorem was restricted to continuous quasiconvex functions.

Theorem 2.17 *([16], Theorem 3.1) Let* $K \subseteq \mathbb{R}^n$ *be nonempty, compact, and convex. If* $g : K \longrightarrow \mathbb{R}$ *is continuous, quasiconvex, and bounded, then there exists a sequence* $\{g_i\}$ *of continuous and strictly quasiconvex functions* $g_i : K \longrightarrow \mathbb{R}$*, such that* $\{g_i\}$ *converges uniformly to* g *on* K*.*

CHAPTER 3

ADJUSTED SUBLEVEL SETS AND THEIR PROPERTIES

In this chapter we will study the adjusted sublevel sets, their continuity properties, and the properties of their normal cones.

3.1 Introduction

The notion of adjusted sublevel set was defined for the first time in [1]. In Definition 2.14 and in [1] two cases were considered, when $x \in \arg\min f$, and when $x \notin \arg\min f$. In fact, if we use the usual convention that $\inf \emptyset = +\infty$, then we see that

$$\operatorname{dist}(x, \emptyset) = +\infty \text{ for all } x \in \mathbb{R}^n,$$

and

$$\bar{B}(\emptyset, +\infty) = \mathbb{R}^n.$$

So, we can write the definition of the adjusted sublevel set without using two cases, as follows.

Definition 3.1 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be any function. The adjusted sublevel set of* f *at* $x \in \mathbb{R}^n$ *is defined by*

$$S_f^a(x) = S_f(x) \cap \bar{B}\left(S_f^<(x), \rho_x^f\right)$$

where

$$\rho_x^f = \operatorname{dist}\left(x, S_f^<(x)\right).$$

Not that for any function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ we have

$$S_f^<(x) \subseteq S_f^a(x) \subseteq S_f(x) \text{ for all } x \in \mathbb{R}^n.$$

Example 3.1 *Let* $f : [-2, 5] \longrightarrow \mathbb{R}$ *be a quasiconvex function defined by*

$$f(x) = \begin{cases} x^2, & -2 \leq x \leq 1, \\ 1, & 1 < x < 3, \\ x - 2, & 3 \leq x \leq 5. \end{cases}$$

The strict sublevel set, the sublevel set, and the adjusted sublevel set of f *at*

$x = 2$ are given as follows:

$$\begin{aligned}
S_f^{<}(2) &=]-1, 1[, \\
S_f(2) &= [-1, 3], \\
\rho_2^f &= 1, \\
\bar{B}\left(S_f^{<}(2), \rho_2^f\right) &= [-2, 2], \\
S_f^{a}(2) &= S_f(2) \cap \bar{B}\left(S_f^{<}(2), \rho_2^f\right) = [-1, 2].
\end{aligned}$$

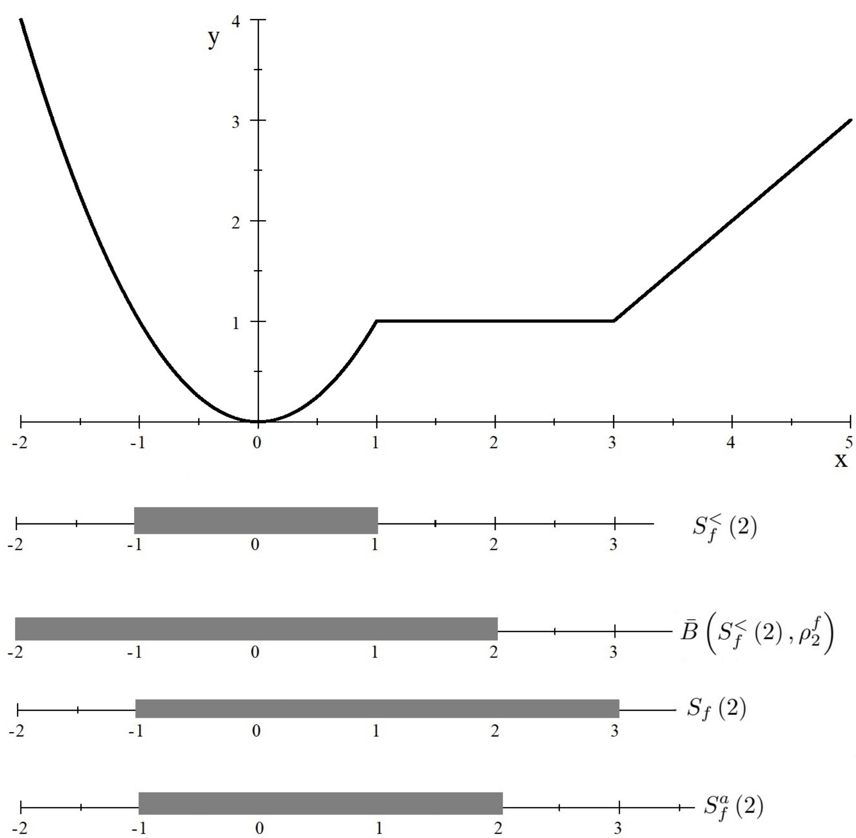

Figure 3.1: Graph and level sets of f

We recall that, a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous if and only if S_λ^f is closed for all $\lambda \in \mathbb{R}$. Thus, if f is l.s.c., then $S_f(x)$ is closed for all $x \in \mathbb{R}^n$. The converse of this implication is not always true, as we can see in the following example.

Example 3.2 *Take f to be the function defined on $\mathbb{R}^n$ by*

$$f(x) = \begin{cases} \| x \|, & \| x \| < 1, \\ \\ 5, & \| x \| \geq 1. \end{cases}, \text{ for every } x \in \mathbb{R}^n.$$

Then $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, while f is not l.s.c. at $\partial B(0,1)$.

The assumption that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$ is usually enough for our purpose, so many times in our work we will use it instead of lower semicontinuity.

The following simple result is very useful for our purposes.

Proposition 3.1 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then $S_f^a(x)$ is closed and convex.*

Proof. Fix $x \in \mathbb{R}^n$. Then $S_f(x)$ is closed by assumption. The quasiconvexity of f implies that $S_f(x)$ and $S_f^<(x)$ are convex.

Define a function $d : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$d(z) = \operatorname{dist}(z, S_f^<(x)).$$

Then

$$S_d(x) = \overline{B}(S_f^<(x), \rho_x^f).$$

But d, being the distance function from a convex set, is known to be convex and Lipschitz continuous.

Hence, $S_d(x) = \overline{B}(S_f^<(x), \rho_x^f)$ is closed and convex.

Thus, $S_f^a(x) = S_f(x) \cap \overline{B}(S_f^<(x), \rho_x^f)$ is closed and convex. ∎

3.2 Continuity Properties of the Adjusted Sublevel Sets

In this section we study the continuity properties of the adjusted sublevel set, as a map $S_f^a : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ where f is our objective function (the study function). We will show that under some natural conditions the map $S_f^a(\cdot)$ is lower semicontinuous. We also show by a counterexample that the map $S_f^a(\cdot)$ is not upper semicontinuous in general.

Our first main result establishes the lower semicontinuity of $S_f^a(\cdot)$.

Theorem 3.3 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $x \rightrightarrows S_f^a(x)$ is l.s.c. on $\mathbb{R}^n$.*

Proof. Fix any $x \in \mathbb{R}^n$ and let $V \subseteq \mathbb{R}^n$ be an open set such that $V \cap S_f^a(x) \neq \emptyset$.

We want to find $\varepsilon > 0$, such that for all $y \in B(x, \varepsilon)$, $V \cap S_f^a(y) \neq \emptyset$.

We consider three cases:

Case (1) $x \in \arg\min f$.

Then for every $y \in \mathbb{R}^n$, $S_f^a(x) = \arg\min f \subseteq S_f^a(y)$ so we have trivially $V \cap S_f^a(y) \neq \emptyset$.

Case (2) $x \notin \arg\min f$ and $\rho_x^f = 0$, i.e., $x \in \overline{S_f^<(x)}$.

Suppose for contradiction that for every $k \in \mathbb{N}$, there exists $y_k \in B(x, \frac{1}{k})$, such that $V \cap S_f^a(y_k) = \emptyset$. Then $V \cap S_f^<(y_k) = \emptyset$, so

$$f(v) \geq f(y_k) \text{ for all } v \in V.$$

Obviously $y_k \longrightarrow x$. But $\{y_k\} \subseteq S_f(v)$ which is closed, so $x \in S_f(v)$. Hence $f(v) \geq f(x)$ for all $v \in V$, which implies $V \cap S_f^<(x) = \emptyset$. This is impossible since the nonempty set $V \cap S_f^a(x)$ is included in $V \cap \overline{B}(S_f^<(x), \rho_x^f) = V \cap \overline{S_f^<(x)}$.

Case (3) $x \notin \arg\min f$ and $\rho_x^f > 0$.

Take any $u \in V \cap S_f^a(x)$. Now take any $w \in S_f^<(x)$ and fix a point v in the open segment joining u and w, close enough to u so that $v \in V$.

Since the function $\operatorname{dist}(\cdot, S_f^<(x))$ is convex and $\operatorname{dist}(u, S_f^<(x)) \leq \rho_x^f$ while $\operatorname{dist}(w, S_f^<(x)) = 0$, we infer that $\operatorname{dist}(v, S_f^<(x)) < \rho_x^f = \operatorname{dist}(x, S_f^<(x))$. Since $\operatorname{dist}(\cdot, S_f^<(x))$ is also continuous, there exists $\varepsilon > 0$ such that for all $y \in B(x, \varepsilon)$, $\operatorname{dist}(v, S_f^<(x)) < \operatorname{dist}(y, S_f^<(x))$. We want to show that $v \in S_f^a(y)$, for all $y \in B(x, \varepsilon)$. Since $v \in V$, this will show that $V \cap S_f^a(y) \neq \emptyset$ for all $y \in B(x, \varepsilon)$ and will prove the theorem.

By quasiconvexity of f we have that $f(v) \leq \max\{f(w), f(u)\} \leq f(x)$.

Now take any $y \in B(x, \varepsilon)$. Then $\operatorname{dist}(y, S_f^<(x)) > \operatorname{dist}(v, S_f^<(x)) \geq 0$ so $y \notin S_f^<(x)$. This means that $f(x) \leq f(y)$, so $v \in S_f(y)$.

If $f(x) < f(y)$, then $v \in S_f^<(y)$, so $v \in S_f^a(y)$.

If $f(x) = f(y)$, then $S_f^<(y) = S_f^<(x)$. Since $\operatorname{dist}(v, S_f^<(x)) < \operatorname{dist}(y, S_f^<(x))$ we get $\operatorname{dist}(v, S_f^<(y)) < \rho_y^f$, so $v \in \overline{B}(S_f^<(y), \rho_y^f)$.

Hence, $v \in S_f^a(y)$ in all cases. ∎

Theorem 3.3 does not hold true if the function f is not quasiconvex, as in the following example.

Example 3.4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function:*

$$f(x) = \begin{cases} 0, & x = (0,0) \\ 1, & x = (t,0)\,, t \in [1,2] \;\; \text{or } x = (0,2) \\ 2, & \text{otherwise.} \end{cases}$$

Then $S_f(x)$ is closed for all x. Let $V = B((0,2)\,,1)$. We set $x_t = (t,0)$ and we check:

$$\text{for all } t \in [1,2], \quad S_f\left(x_t\right) = \{(0,0)\} \cup ([1,2] \times \{0\}) \cup \{(0,2)\}$$

$$\text{for all } t \in [1,2], \quad S_f^<\left(x_t\right) = \{(0,0)\}$$

$$\text{for all } t \in [1,2], \quad \rho_{x_t}^f = t$$

$$\text{for all } t \in [1,2], \quad \overline{B}\left(S_f^<\left(x_t\right), \rho_{x_t}^f\right) = \overline{B((0,0), t)}$$

$$\text{for all } t \in [1,2[, \quad S_f^a(x_t) = \{(0,0)\} \cup ([1,t] \times \{0\})$$

$$\text{for } t = 2, \quad S_f^a(x_2) = \{(0,0)\} \cup ([1,2] \times \{0\}) \cup \{(0,2)\}$$

We see that $S_f^a(x_2) \cap V = \{(0,2)\} \neq \emptyset$, but for each $t \in [1,2[, \quad S_f^a(x_t) \cap V = \emptyset$. Taking $x_{2-\frac{1}{k}}$, we have a counterexample to Case 3. In fact, the map $x \rightrightarrows S_f^a(x)$ is not l.s.c. at x_2.

Theorem 3.3 does not hold, if the lower level sets are not closed, even if f is quasiconvex, as we can see in the following example.

Example 3.5 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$f(x) = \begin{cases} x^2, & x \in]-2, 6[, \\ \\ +\infty, & x \notin]-2, 6[. \end{cases}$$

Then

$$S_f^a(-2) = [-2, 6]$$

Take $V =]5, 7[$. *Then*

$$V \cap S_f^a(-2) \neq \emptyset$$

Let $y_k = -2 + \frac{1}{k}$, $k \in \mathbb{N}$, *then*

$$y_k \longrightarrow -2$$

But

$$S_f^a(y_k) \cap V = \left[-2 + \frac{1}{k}, 2 - \frac{1}{k}\right] \cap]5, 7[= \emptyset, \textit{ for all } k \in \mathbb{N}$$

Hence, S_f^a *does not satisfy the l.s.c. at* $x = -2$.

Note that, f *is quasiconvex, but* $S_f(2) =]-2, 2]$, *which is not closed.*

We can follow the proof of Case (2) in Theorem 3.3 to prove the lower semi-continuity of the strict sublevel set map for any real-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$.

Theorem 3.6 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any function. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $x \rightrightarrows S_f^<(x)$ is l.s.c. on $\mathbb{R}^n$.*

Proof. Fix any $x \in \mathbb{R}^n$, and let $V \subseteq \mathbb{R}^n$ be an open set, such that $V \cap S_f^<(x) \neq \emptyset$.

We want to find $\varepsilon > 0$ such that

$$V \cap S_f^<(y) \neq \emptyset, \text{ for all } y \in B(x, \varepsilon).$$

Assume for contradiction, that for all $k \in \mathbb{N}$ there exists $y_k \in B\left(x, \frac{1}{k}\right)$ such that $V \cap S_f^<(y_k) = \emptyset$. This implies

$$f(v) \geq f(y_k) \text{ for all } (k, v) \in \mathbb{N} \times V.$$

But $y_k \longrightarrow x$, and $S_f(v)$ is closed for all $v \in V$.

Hence,

$$x \in S_f(v) \text{ for all } v \in V,$$

so

$$f(x) \leq f(v) \text{ for all } v \in V.$$

This means

$$v \notin S_f^<(x) \text{ for all } v \in V,$$

which is a contradiction to the assumption, that $S_f^<(x) \cap V \neq \emptyset$. ∎

The last theorem is very close to ([2], Lemma 1).

Theorem 3.7 *([2], Lemma 1) If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is l.s.c., then the map $\bar{S}_f^{<}(\cdot)$ is l.s.c. on $\mathbb{R}^n$.*

The map $S_f^a(\cdot)$ is not always u.s.c., even if our function f is quasiconvex and continuous, as seen in the following example.

Example 3.8 *Take the function f to be the same as ([4], Example 2.2); $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by*

$$f(x,y) = \begin{cases} \max\{x,y\}, & x < 0 \textit{ and } y < 0, \\ 0, & x \geq 0 \textit{ and } y < 0, \\ y, & \textit{elsewhere.} \end{cases}$$

Then

$$S_f^a(0,0) = \{(p,q), p,q \leq 0\},$$

and

$$S_f^a(x,y) = \{(t,r) : r \leq y, t \in \mathbb{R}\} \textit{ for every } y > 0.$$

Now take $V = \{(x,y) : x,y < 1\}$ which is an open neighborhood of $S_f^a(0,0)$.

For all U open with $(0,0) \in U$ there exist $(x,y) \in U$ such that $y > 0$ and so $S_f^a(x,y) \nsubseteq V$. This implies S_f^a is not u.s.c..

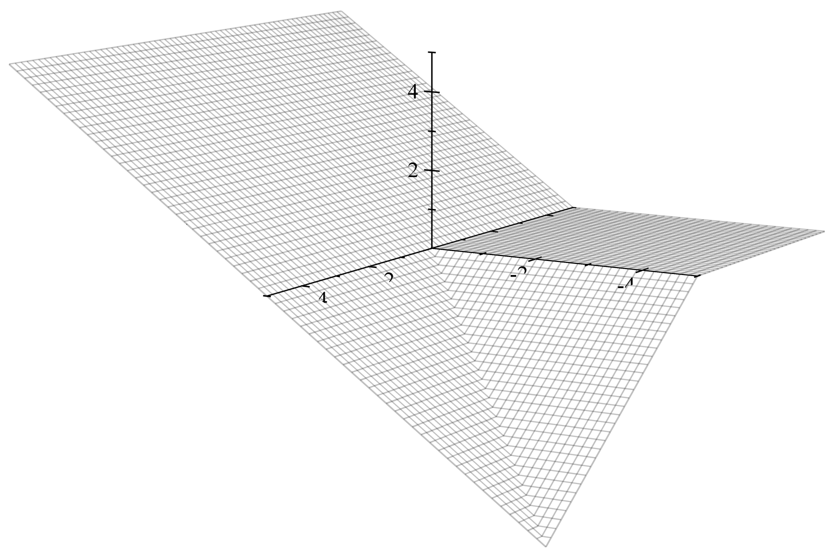

Figure 3.2: Graph of $f(x, y)$

3.3 Properties of the Normal Cones

Upper semicontinuity does not fit well with cone-valued maps. See for instance Example 2.3 in [4], and Proposition 2.1(i) in [13]. For this reason, we will use the so-called cone upper semicontinuity. Before we state its definition, we recall the definition of conic neighborhood of a cone $L \subseteq \mathbb{R}^n$.

Definition 3.2 *A conic neighborhood of a cone $L \subseteq \mathbb{R}^n$ is an open cone $M \subseteq \mathbb{R}^n$, such that $L \subseteq M \cup \{0\}$.*

Definition 3.3 *A cone-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be cone upper semicontinuous at $x \in \mathbb{R}^n$, if for every conic neighborhood M of $T(x)$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$, such that $x \in U$ and M is a conic neighborhood of $T(u)$ for any $u \in U$.*

It is not always easy to prove the cone upper semicontinuity of a map by definition. In the next theorem we will show an equivalent continuity condition on the map that is produced from the intersection of the cone-valued map and the unit sphere.

Given a multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we recall that the domain of T is the set D(T)$= \{x \in \mathbb{R}^n : T(x) \neq \phi\}$. We set

$$S(0,1) = \{x \in \mathbb{R}^n : \|x\| = 1\}\,.$$

Theorem 3.9 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map with closed values. Then T is cone u.s.c. at $x \in \mathrm{D}(T)$ if and only if the map $F(\cdot) = T(\cdot) \cap S(0,1)$ is u.s.c. at x.*

Proof. Assume that T is cone u.s.c.. Since $S(0,1)$ is compact, in order to show that F is u.s.c., it is enough to show that it has closed graph [11, Proposition 2.23].

So let $(x_k, y_k) \in \operatorname{Graph} F$, $k \in \mathbb{N}$, be such that $(x_k, y_k) \to (x, y)$.

Assume that $y \notin F(x)$; then $y \notin T(x)$. Since $T(x)$ is closed, there exists $\delta \in]0,1[$ such that $\overline{B}(y,\delta) \cap T(x) = \emptyset$. Let $K = \mathrm{cone}(\overline{B}(y,\delta))$. Then $K \cap T(x) = \{0\}$. The set K^c is an open cone, and $T(x) \subseteq K^c \cup \{0\}$. Consequently, there exists $\varepsilon > 0$ such that for all $x' \in B(x,\varepsilon)$, $T(x') \subseteq K^c \cup \{0\}$.

For k sufficiently large, one has $x_k \in B(x,\varepsilon)$. Thus, $y_k \in T(x_k) \subseteq K^c \cup \{0\}$. It follows that $y_k \notin \overline{B}(y,\delta)$. This contradicts $y_k \to y$, and proves that $\operatorname{Graph} F$ is closed.

Conversely, let F be u.s.c. at x, and M be a conic neighborhood of $T(x)$ (i.e., $T(x) \subseteq M \cup \{0\}$). Then M is a neighborhood of $F(x)$, so there exists a neighborhood of x (say, U) such that

$$F(u) \subseteq M, \text{ for all } u \in U.$$

Hence

$$T(u) \cap S(0,1) \subseteq M.$$

But M and $T(u)$ are cones, so

$$T(u) \subseteq M \cup \{0\}, \text{ for all } u \in U.$$

Thus, T is cone u.s.c. at x. ∎

The equivalence of the cone upper semicontinuity of T with the upper semicontinuity of F, established in Theorem 3.9, was also stated without proof in [4].

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be cone valued map, α, β be two positive real numbers such that $0 < \alpha \leq \beta$. Then we define $T_{\alpha,\beta}$ at any $x \in \operatorname{dom} T$ by

$$T_{\alpha,\beta}(x) = \{z \in T(x) : \alpha \leq z \leq \beta\}.$$

Proposition 3.2 *(See [4]) Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone valued map. Then T is cone upper semicontinuous at $x \in \mathbb{R}^n$ if and only if $T_{\alpha,\beta}$ is u.s.c. at x.*

The lower semicontinuity of a set-valued map implies the cone upper semicontinuity of its "normal cone operator", as we now show.

Theorem 3.10 *Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a l.s.c. map. Let further $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map defined by*

$$M(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in A(x)\}, \quad x \in \mathbb{R}^n.$$

Then M is cone u.s.c. on $\mathbb{R}^n$.

Proof. Fix $x \in \mathbb{R}^n$. It is enough to show that $F(\cdot) = M(\cdot) \cap S(0,1)$ is u.s.c. at x.

Suppose for contradiction that F is not u.s.c. at x. Then there exists an open neighborhood of $F(x)$ (say $V \subseteq \mathbb{R}^n$), such that for all $k > 0$, there exist $x_k \in B(x, \frac{1}{k})$ and $y_k \in F(x_k) \cap V^c$. Without loss of generality assume that $y_k \longrightarrow y$ for some $y \in \mathbb{R}^n$. Now $\langle y_k, z - x_k\rangle \leq 0$ for all $z \in A(x_k)$. But A is l.s.c., so Proposition 2.6 in ([11], Chapter 1) says that, for any $q \in A(x)$, there exists $z_k \in A(x_k)$ such that $z_k \longrightarrow q$. Hence $\langle y_k, z_k - x_k\rangle \leq 0$. This implies that $\lim \langle y_k, z_k - x_k\rangle \leq 0$, so $\langle y, q - x\rangle \leq 0$, where $q \in A(x)$ is arbitrary. Consequently, $y \in M(x)$. In addition, $y_k \in F(x_k) \subseteq S(0,1)$ entails $y \in S(0,1)$. So $y \in M(x) \cap S(0,1) = F(x) \subseteq V$. On the other hand, $y_k \in V^c$ and $y = \lim y_k$ imply $y \in V^c$ because V^c is closed. We arrived to a contradiction, hence F must be u.s.c. at x, and so $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a cone u.s.c. map. ∎

We provide an alternative proof of Theorem 3.10, using the following result of

Aussel and Pistek.

Proposition 3.3 *[2, Lemma 1] Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a l.s.c. map. Let further $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map defined by*

$$M(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in A(x)\}, \quad x \in \mathbb{R}^n.$$

Then M is closed.

Proof of Theorem 3.10. By Proposition 3.3, the map M has a closed graph. It follows easily that the map $F(\cdot) = M(\cdot) \cap S(0,1)$ also has a closed graph. Indeed, if $(x_k, y_k) \longrightarrow (x, y)$ with $y_k \in F(x_k)$ for all $k \in \mathbb{N}$, we can say that

$$y_k \in M(x_k) \cap S(0,1),$$

from the closeness of M and $S(0,1)$, we have

$$y \in M(x) \cap S(0,1) = F(x).$$

Using again [11, Proposition 2.23], we deduce that F is u.s.c.. Thus, M is cone u.s.c., in view of Theorem 3.9. ■

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any function. Then for any $x \in \mathbb{R}^n$, the normal cone to the adjusted sublevel set $S_f^a(x)$ at x, is by definition the set

$$N_f^a(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in S_f^a(x)\}.$$

The normal cone to the strict sublevel set $S_f^<(x)$ at x, is the set

$$N_f^<(x) = \{y \in \mathbb{R}^n : \langle y, z - x\rangle \leq 0 \text{ for all } z \in S_f^<(x)\}.$$

We are ready now to state our corollaries regarding the cone upper semicontinuity of the normal cone maps.

Corollary 3.11 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $N^a(\cdot)$ is cone u.s.c. on $\mathbb{R}^n$.*

Proof. Theorem 3.3 implies that $S_f^a(\cdot)$ is l.s.c. on $\mathbb{R}^n$. Then by Theorem 3.10 we can say that $N_f^a(\cdot)$ is cone u.s.c.. ▮

The last corollary recovers Proposition 3.5 of [1] in the finite-dimensional case, without using any assumption on $\text{int} S_f^a(x)$.

Corollary 3.12 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any function. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $N^<(\cdot)$ is cone u.s.c. on $\mathbb{R}^n$.*

Proof. Theorem 3.6 implies that $S_f^<(\cdot)$ is l.s.c. on $\mathbb{R}^n$. Then by Theorem 3.10 we can say that $N_f^<(\cdot)$ is cone u.s.c. on $\mathbb{R}^n$. ▮

Corollary 3.12 is a generalization of Proposition 2.2 of [4].

Just as with the usual upper semicontinuity, cone upper semicontinuity of a map T with closed values, implies that T is closed.

Proposition 3.4 *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a cone-valued map with closed values. If T is cone u.s.c., then T is closed.*

Proof. Since T is cone u.s.c., by Theorem 3.9 the map $F(.) = T(.) \cap S(0,1)$ is u.s.c. with closed values. Thus it is closed [11, p. 41].

Let $\{(x_k, y_k)\}$ be a sequence in Graph(T), such that $(x_k, y_k) \longrightarrow (x, y)$.

If $y = 0$, then trivially $y \in T(x)$ because $T(x)$ is a closed cone, so it contains 0.

If $y \neq 0$, then $\{(x_k, \frac{y_k}{\|y_k\|})\} \subseteq \text{Graph}(F)$ for large k, and $(x_k, \frac{y_k}{\|y_k\|}) \longrightarrow (x, \frac{y}{\|y\|})$. Hence $\frac{y}{\|y\|} \in F(x)$. So $\frac{y}{\|y\|} \in T(x)$, and since $T(x)$ is a cone, $y \in T(x)$.

Thus $y \in T(x)$ in both cases, which implies that T is closed. ▮

Now we can apply Proposition 3.4, and the last two corollaries to prove the closeness of $N_f^a(\cdot)$, and $N_f^<(\cdot)$.

Corollary 3.13 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be quasiconvex. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $N^a(\cdot)$ is closed.*

Corollary 3.14 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be any function. If $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, then the map $N^<(\cdot)$ is closed.*

CHAPTER 4

DECOMPOSITION OF QUASICONVEX FUNCTIONS

Our objective in this chapter is to write any l.s.c. quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ as a composition

$$f = h \circ g,$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a quasiconvex function which does not have any local minimum except the global minimum, and $h : \mathrm{Im}(g) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is nondecreasing.

4.1 Introduction

Semistrictly quasiconvex functions are known to have the property that every local minimum is global. However, the following example shows that there exist quasiconvex functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ for which there is no decomposition

$$f = h \circ g,$$

where g is semistrictly quasiconvex, and $h : \mathrm{Im}\,(g) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is non decreasing.

Example 4.1 *Take f to be a function defined on the positive orthant $\mathbb{R}^2_+$ in polar coordinates by*

$$f\,(r, \theta) = \begin{cases} \theta, & r > 0, \\ 0, & r = 0. \end{cases}$$

f is set to be equal to $+\infty$ outside the positive orthant.

This function is quasiconvex and l.s.c.. Assume that

$$f = h \circ g,$$

where g is semistrictly quasiconvex and h is nondecreasing.

Choose $0 < r_1 < r_2$.

For each $\theta \in \left]0, \frac{\pi}{2}\right[$ one has

$$f(0, \theta) < f(r_2, \theta),$$

so

$$g(0, \theta) < g(r_2, \theta).$$

By semistrict quasiconvexity,

$$g(r_1, \theta) < g(r_2, \theta).$$

Note also that for $\theta \neq \theta'$ *the intervals* $[g(r_1, \theta), g(r_2, \theta)]$ *and* $[g(r_1, \theta'), g(r_2, \theta')]$ *are disjoint since if, say,* $\theta < \theta'$ *then* $f(r_2, \theta) < f(r_1, \theta')$ *so* $g(r_2, \theta) < g(r_1, \theta')$. *Thus we have an uncountable set of disjoint nondegenerate intervals* $[g(r_1, \theta), g(r_2, \theta)]$, $\theta \in \left]0, \frac{\pi}{2}\right[$. *Now for each* $\theta \in \left]0, \frac{\pi}{2}\right[$, *choose a rational number* $q_\theta \in [g(r_1, \theta), g(r_2, \theta)]$. *Since the intervals* $[g(r_1, \theta), g(r_2, \theta)]$ *are disjoint for different values of* θ, *the numbers* q_θ *are different for different values of* θ. *Thus, the set*

$$\left\{q_\theta : \theta \in \left]0, \frac{\pi}{2}\right[\right\}$$

is uncountable. We arrived to a contradiction, since the set of all rational numbers is countable.

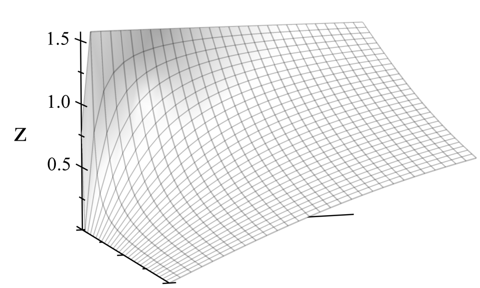

Figure 4.1: Graph of $f(r, \theta)$

We have to replace semistrict quasiconvexity by a weaker notion. As the example shows, we cannot avoid g having on a segment a constant value greater than $\arg\min g$. But we can avoid having an n-dimensional "flat" part on its graph, thus escaping the main inconvenience of quasiconvexity. This will be done through a generalization of the notion of g-pseudoconvexity of Crouzeix et al.

On the other hand, we may replace lower semicontinuity by the weaker assumption that for each $x \in \mathbb{R}^n$, $S_f(x)$ is closed.

According to [6], a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is called g-pseudoconvex (g stands for geometrically) if it is quasiconvex and for every $x \in \mathbb{R}^n$ with $f(x) > \inf f$, $\operatorname{int} S_f(x) \neq \emptyset$ holds, and the sets $S_f(x)$ and $S_f^<(x)$ have the same interior and the same closure. See also [7].

In search of a more general definition, take f to be quasiconvex. Recall the following properties.

Proposition 4.1 *([15, Proposition 1.73]) Let $K \subseteq \mathbb{R}^n$ be nonempty and convex. Then*

i) $\overline{\mathrm{ri} K} = \bar{K}$.

ii) $\mathrm{ri} K = \mathrm{ri} \bar{K}$.

By Proposition 4.1 we can prove the following corollary.

Corollary 4.2 *Let A and B be two convex subsets of $\mathbb{R}^n$. Then*

i) *If A and B have the same closure, then they have the same relative interior.*

ii) *If A and B have the same relative interior, then they have the same closure.*

Proof.

i) By applying Part (ii) of Proposition 4.1, we have

$$\mathrm{ri}A = \mathrm{ri}\bar{A} = \mathrm{ri}\bar{B} = \mathrm{ri}B.$$

ii) By applying Part (i) of Proposition 4.1, we have

$$\bar{A} = \overline{\mathrm{ri}A} = \overline{\mathrm{ri}B} = \overline{B}.$$

∎

Now we adopt the definition.

Definition 4.1 *A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called neatly quasiconvex if it is quasiconvex and for every x with $f(x) > \inf f$, the sets $S_f(x)$ and $S_f^{<}(x)$ have the same closure (or equivalently, the same relative interior).*

It is clear from the definition that a g-quasiconvex function is also neatly quasiconvex. The following example shows that the converse is not true.

Example 4.3 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by*

$$f(x, y) = \begin{cases} -e^{-|x|}, & y = 0, \\ |y|, & y \neq 0. \end{cases}$$

The only local minimum of f is at $(x, y) = (0, 0)$, i.e., the only local minimum is global minimum. By Proposition 4.2 that follows, this implies that f is neatly quasiconvex. On the other hand, f fails to be g-pseudoconvex, since $\mathrm{int}S_f(x, y) = \emptyset$ *for some* $(x, y) \notin \arg\min f$.

In the next proposition we will prove a condition which is equivalent to the neat quasiconvexity.

Proposition 4.2 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quasiconvex function. Then the following are equivalent:*

(a) f is neatly quasiconvex

(b) For each $x \notin \arg\min f$, $\rho_x^f = 0$

(c) Each local minimum of f is a global minimum.

Proof.

a $\Longrightarrow$ b: If f is neatly quasiconvex, then for each $x \notin \arg\min f$, $x \in \overline{S_f(x)} = \overline{S_f^<(x)}$ so $\rho_x^f = 0$.

b $\Longrightarrow$ c: Assume that $\bar{x} \notin \arg\min(f)$. By assumption $\rho_x^f = 0$, so $\bar{x} \in \overline{S_f^<(\bar{x})}$. Hence, every ball $B(\bar{x}, \varepsilon)$ intersects $S_f^<(\bar{x})$, which implies that $\bar{x}$ is not a local minimum.

c $\Longrightarrow$ a: If f is quasiconvex, but not neatly quasiconvex, then there exists x with $f(x) > \inf f$ such that $\overline{S_f(x)} \setminus \overline{S_f^<(x)} \neq \phi$. Then there exists

$\bar{x} \in S_f(x) \setminus \overline{S_f^<(x)}$ (otherwise, $S_f(x) \subseteq \overline{S_f^<(x)}$, this implies $\overline{S_f(x)} \subseteq \overline{S_f^<(x)}$).

Then $f(\bar{x}) = f(x)$ and $\bar{x} \notin \overline{S_f^<(x)} = \overline{S_f^<(\bar{x})}$, so there exists $\varepsilon > 0$ such that

$$f(y) \geq f(\bar{x}) \text{ for all } y \in B(\bar{x}, \varepsilon).$$

Hence, $\bar{x}$ is a local minimum.

But, from the assumption we have

$$f(\bar{x}) = f(x) > \inf f,$$

so $\bar{x}$ is not a global minimum. Then $\bar{x}$ is a local minimum, not global , which is a contradiction to the assumption (c).

∎

4.2 The Class of the Adjusted Sublevel Sets

In this section we assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$. We denote the class of all adjusted sublevel sets of f by $\mathcal{C}$.

$$\mathcal{C} = \left\{ S_f^a(x) : x \in \mathbb{R}^n \right\}.$$

The class $\mathcal{C}$ has the following properties:

Proposition 4.3 *$\mathcal{C}$ is totally ordered by inclusion. That is, for every $A, B \in \mathcal{C}$, $A \subseteq B$ or $B \subseteq A$ holds.*

Proof. Let $x, y \in \mathbb{R}^n$. Then we have the following cases:

i) If $f(x) < f(y)$, then

$$S_f^a(x) \subseteq S_f(x) \subseteq S_f^<(y) \subseteq S_f^a(y).$$

Since $y \in S_f^a(y)$ but $y \notin S_f(x) \supseteq S_f^a(x)$, we obtain

$$S_f^a(x) \subsetneq S_f^a(y).$$

The case $f(y) < f(x)$ is similar.

ii) If $f(y) = f(x)$, and $\rho_x^f = \rho_y^f$, then

$$S_f(x) = S_f(y),$$

and

$$S_f^<(x) = S_f^<(y).$$

But

$$\rho_x^f = \rho_y^f,$$

implies

$$\overline{B}(S_f^<(x), \rho_x^f) = \overline{B}(S_f^<(y), \rho_y^f).$$

Hence

$$S_f^a(x) = S_f^a(y).$$

iii) If $f(y) = f(x)$, and $\rho_x^f < \rho_y^f$, then

$$S_f(x) = S_f(y),$$

and

$$S_f^{<}(x) = S_f^{<}(y).$$

But

$$\rho_x^f < \rho_y^f,$$

implies

$$\overline{B}(S_f^{<}(x), \rho_x^f) \subseteq \overline{B}(S_f^{<}(y), \rho_y^f).$$

Hence

$$S_f^a(x) \subseteq S_f^a(y).$$

Since $y \in S_f^a(y)$, but

$$y \notin \overline{B}(S_f^{<}(x), \rho_x^f) \supseteq S_f^a(x),$$

we get

$$S_f^a(x) \subsetneq S_f^a(y).$$

The case $f(y) = f(x)$ and $\rho_y^f < \rho_x^f$ is similar.

∎

In fact, the above proof shows that for $x, y \in \mathbb{R}^n$, one has $S_f^a(x) \subsetneq S_f^a(y)$ if and only if either $f(x) < f(y)$, or $f(x) = f(y)$ and $\rho_x^f < \rho_y^f$; $S_f^a(x) = S_f^a(y)$ if and only if $f(x) = f(y)$ and $\rho_x^f = \rho_y^f$.

Proposition 4.4 *Let $x, y \in \mathbb{R}^n$. Then $x \in S_f^a(y)$ if and only if $S_f^a(x) \subseteq S_f^a(y)$.*

Proof. Assume that $x \in S_f^a(y)$, this implies $x \in S_f(y)$. Hence, we have two cases:

i) $f(x) < f(y)$, then $S_f^a(x) \subsetneq S_f^a(y)$.

ii) $f(x) = f(y)$, then

$$x \in S_f^a(y) \subseteq \overline{B}(S_f^<(y), \rho_y^f) = \overline{B}(S_f^<(x), \rho_y^f),$$

so

$$\rho_x^f \leq \rho_y^f,$$

and

$$S_f^a(x) \subseteq S_f^a(y).$$

The converse is obvious since $x \in S_f^a(x) \subseteq S_f^a(y)$.

∎

Proposition 4.5 *For every $x \in \mathbb{R}^n \backslash \arg\min f$, $x \in \partial S_f^a(x)$.*

Proof. Take $x \in \mathbb{R}^n \backslash \arg\min f$. Then

$$\overline{B}(S_f^<(x), \rho_x^f) \neq \emptyset.$$

Since $x \notin S_f^<(x)$, either $\rho_x^f = 0$ or $\rho_x^f > 0$.

If $\rho_x^f = 0$, then x belongs to the boundary of $S_f^<(x)$.

If $\rho_x^f > 0$, then again x belongs to the boundary of $\overline{B}(S_f^<(x), \rho_x^f)$.

In both cases, the property follows. ∎

Proposition 4.6 *If $A, B \in \mathcal{C}$, $A \subsetneq B$ and $\mathrm{int}B \neq \emptyset$, then $(\mathrm{int}B)\backslash A \neq \emptyset$. More generally, if $A \subsetneq B$, then also $(\mathrm{ri}B)\backslash A \neq \emptyset$.*

Proof. Take $A, B \in \mathcal{C}$, $A \subsetneq B$, then there exist $x, y \in \mathbb{R}^n$ such that

$$A = S_f^a(x) \text{ and } B = S_f^a(y).$$

For any $z \in \mathrm{ri}B$, $]y, z[\subseteq \mathrm{ri}B$. Since A is closed and $y \notin A$ (if $y \in S_f^a(x)$ then $S_f^a(y) \subseteq S_f^a(x)$ by Proposition 4.4 above), we can take $w \in]y, z[$ sufficiently close to y so that $w \notin A$. This proves that $\mathrm{ri}B\backslash A \neq \emptyset$. ∎

For any $m \in \{0, 1, 2, \ldots, n\}$, we define $\mathcal{C}_m$ to be the class of all adjusted sublevel sets with dimension m.

$$\mathcal{C}_m = \{A \in \mathcal{C} : \dim A = m\}, \; m = 0, 1, 2, \ldots, n.$$

Proposition 4.7 *All elements of $\mathcal{C}_m$ generate the same affine subspace of dimention m, $m = 0, 1, 2, \ldots, n$.*

Proof. Assume that $A, B \in \mathcal{C}_m$.

If $A = B$, then

$$\mathrm{aff}A = \mathrm{aff}B.$$

If $A \subsetneq B$, then

$$\text{aff}A \subseteq \text{aff}B.$$

But

$$\dim(\text{aff}A) = \dim A = \dim B = \dim(\text{aff}B).$$

Hence,

$$\text{aff}A = \text{aff}B.$$

$\blacksquare$

4.3 The Main Result

In this section we will use the results from the previous section, to construct the decomposition of a quasiconvex function with closed sublevel sets.

Theorem 4.4 *For every quasiconvex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $S_f(x)$ is closed for all $x \in \mathbb{R}^n$, there exists a neatly quasiconvex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $S_g(x) = S_f^a(x)$ for all $x \in \mathbb{R}^n$, and a nondecreasing function $h : \text{Im}g \to \mathbb{R} \cup \{+\infty\}$ such that $f = h \circ g$.*

Proof. The proof is divided into seven steps as follows:

- Step(1): We construct a function $k : \mathcal{C} \to \mathbb{R}$.

  Assume first that all elements of $\mathcal{C}$ have nonempty interior. Let α be a continuous, positive function on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} \alpha(x) d\mu = 1, \ (\mu \text{ the Lebesgue measure}).$$

For each $A \in \mathcal{C}$, set

$$k(A) = \int_A \alpha(x) d\mu + n - 1.$$

By Proposition 4.6 above, k is increasing, i.e., if $A, B \in \mathcal{C}$, $A \subsetneq B$ then

$$n - 1 < k(A) < k(B) \leq n.$$

In the general case, we write

$$\mathcal{C}_m = \{A \in \mathcal{C} : \dim A = m\}, \ 0 \leq m \leq n.$$

Then by Proposition 4.7, all elements of $\mathcal{C}_m$ generate the same affine subspace V_m of dimension m, and have a nonempty interior with respect to V_m. We define an increasing function k_m on $\mathcal{C}_m$, $1 \leq m \leq n$ exactly as before (we only need Propositions 4.3 and 4.6) and set $k_0(A) = 0$ for the unique element of $\mathcal{C}_0$ (if it exists).

Finally we define k on $\mathcal{C}$ by

$$k(A) = k_m(A) \text{ if } A \in \mathcal{C}_m.$$

- Step(2): We show that the function k is increasing (i.e, if $A, B \in \mathcal{C}$ and $A \subsetneq B$, then $k(A) < k(B)$).

  Before we prove this property, note that

$$\mathrm{ri}\,(B) \supseteq \mathrm{ri}\,(B \backslash A) \neq \emptyset.$$

  Now, if $\dim(A) = \dim(B) = m$, then

$$k(A) = k_m(A) = \int_A \alpha(x) d\mu + m - 1 < \int_B \alpha(x) d\mu + m - 1 = k_m(B) = k(B).$$

  If $\dim(A) = m < p = \dim(B)$, then

$$k(A) = k_m(A) = \int_A \alpha(x) d\mu + m - 1 \leq m \leq p - 1 < p - 1 + \int_B \alpha(x) d\mu = k_p(B) = k(B).$$

  Hence, k is increasing.

- Step(3): We define a function $g : \mathbb{R}^n \to \mathbb{R}$ such that its lower level sets are the elements of $\mathcal{C}$ as follows. Set for each $x \in \mathbb{R}^n$

$$g(x) = k(S_f^a(x)). \tag{4.1}$$

  Let us show that $S_g(x) = S_f^a(x)$.

  Indeed, $y \in S_g(x)$ iff $g(y) \leq g(x)$ or $k(S_f^a(y)) \leq k(S_f^a(x))$. Since $\mathcal{C}$ is totally ordered and k is increasing, we obtain that $S_f^a(y) \subseteq S_f^a(x)$. By Proposition

4.4, this happens exactly when $y \in S_f^a(x)$.

It follows that g is quasiconvex and such that $S_g(x)$ are closed for all $x \in \mathbb{R}^n$.

- Step(4): We show that the function g defined by (4.1) is neatly quasiconvex.

Indeed, according to Proposition 4.2, to check that g is neatly quasiconvex, it is enough to show that for every x, $\rho_x^g = 0$. To see this, assume first that $\rho_x^f > 0$. Let y be the projection of x onto $\overline{S_f^<(x)}$. For each $z \in]x, y[$ we have

$$z \in S_f(x) \backslash S_f^<(x),$$

so

$$f(z) = f(x).$$

Thus

$$S_f^<(x) = S_f^<(z),$$

and we deduce that

$$\rho_x^f = d(x, \overline{S_f^<(x)}) > d(z, \overline{S_f^<(z)}) = \rho_z^f.$$

Hence, $x \notin S_f^a(z)$. Thus $S_f^a(z) \subsetneq S_f^a(x)$ so $g(z) < g(x)$ by (4.1) since k is strictly increasing.

Hence $]x, y[\subseteq S_g^<(x)$ so $x \in \overline{S_g^<(x)}$, i.e., $\rho_x^g = 0$.

Now assume that $\rho_x^f = 0$. Then $x \in \overline{S_f^<(x)}$ so there exists a sequence $\{x_k\}$

such that $f(x_k) < f(x)$ and $x_k \to x$.

Since $x \notin S_f(x_k)$, we also have

$$x \notin S_f^a(x_k).$$

This implies $g(x) > g(x_k)$. Thus,

$$x_k \in S_g^<(x),$$

so

$$x \in \overline{S_g^<(x),}$$

and $\rho_x^g = 0$ also in this case.

- Step(5): We construct the function h.

First we note that for each $t \in \operatorname{Im} g$, there exists $x \in \mathbb{R}^n$ such that $g(x) = t$. Define the function $h : \operatorname{Im} g \to \mathbb{R} \cup \{+\infty\}$ by

$$h(t) = f(x).$$

Let us show that the function h is well-defined. For this, we need to show that for each $t \in \operatorname{Im} g$, if $t = g(x)$, then the value $f(x)$ does not depend on x but only depends on t. Indeed, let $x_1, x_2 \in \mathbb{R}^n$ be such that $g(x_1) = t = g(x_2)$. Then $k\left(S_f^a(x_1)\right) = k\left(S_f^a(x_2)\right)$. By Step (2) we have $S_f^a(x_1) = S_f^a(x_2)$. This implies $x_1 \in S_f^a(x_2)$ and $x_2 \in S_f^a(x_1)$. Thus, $f(x_1) = f(x_2)$.

That is, the value of f only depends on t, so h is well-defined.

- Step(6): We show that h is nondecreasing on $\operatorname{Im}(g)$.

  Assume that $t_1, t_2 \in \operatorname{Im}(g)$ such that $t_1 < t_2$, then there exist $x_1, x_2 \in \mathbb{R}^n$ such that

$$t_1 = g(x_1) \text{ and } t_2 = g(x_2).$$

We want to show that

$$h(t_1) \leq h(t_2),$$

or equivalently

$$f(x_1) \leq f(x_2).$$

Indeed, since

$$g(x_1) = t_1 < t_2 = g(x_2),$$

we can say

$$k\left(S_f^a(x_1)\right) < k\left(S_f^a(x_2)\right).$$

Then from Step (2), we have

$$S_f^a(x_1) \subsetneq S_f^a(x_2).$$

This implies $x_1 \in S_f^a(x_2)$, so we have

$$f(x_1) \leq f(x_2),$$

or

$$h(t_1) \leq h(t_2).$$

Hence, $h : \mathrm{Im}(g) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is nondecreasing.

- Step(7): It is clear that

$$h(g(x)) = f(x) \text{ for all } x \in \mathbb{R}^n,$$

i.e.,

$$f = h \circ g.$$

∎

In the previous theorem we assumed that for each $x \in \mathbb{R}^n$, the set $S_f(x)$ is closed. Since $S_g(x) = S_f^a(x)$, it follows from Proposition 3.1 that the function g constructed in the theorem still has this property. The question whether g can be chosen semicontinuous when f is lower semicontinuous is still open, and possibly requires a different or modified approach.

We give below an example to show the construction of the function g, as in the theorem.

Example 4.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by*

$$f(x) = \begin{cases} -1 + \sqrt{-x}, & x \leq 0 \\ n, & n < x \leq n+1, \quad n = 0, 1, 2, \ldots \end{cases}$$

We choose

$$\alpha(x) = \frac{1}{\pi\left(1+x^2\right)}.$$

The following table gives $S_f^a(x)$ and $g(x)$ for all possible values of x. Here, $n = 0, 1, 2, \ldots$.

x	$S_f^a(x)$	$g(x) = \int_{S_f^a(x)} \alpha(x)dx$
$-\left(n+1\right)^2 \leq x < -n^2$	$[x, n]$	$\frac{1}{\pi}\left(\arctan n - \arctan x\right)$
$x = 0$	$\{0\}$	0
$n < x \leq n+1$	$\left[-\left(n+1\right)^2, x\right]$	$\frac{1}{\pi}\left(\arctan x + \arctan\left(n+1\right)^2\right)$

Table 4.1: $S_f^a(x)$ and $g(x)$

One can see that f has discontinuities in the positive real axis, whereas g has discontinuities in the positive and in the negative axis. This is not due to the particular construction used in Theorem 4.4; it would be present in any decomposition $f = h \circ g$ with h nondecreasing and g neatly quasiconvex. Indeed, in the positive axis the function f has some intervals where it is constant. If, say, f is constant on the interval $[a, b]$, $0 < a < b$, then $g(0) < g(a) < g(b)$ and h has to be constant on the interval $[g(a), g(b)]$. This means that for x in the negative axis, the range of g cannot contain $[g(a), g(b)]$ because otherwise f would be constant on some interval. Thus, necessarily g is discontinuous on $]-\infty, 0[$. An even more extreme behavior can be seen in Example 4.6 below, where g is necessarily discontinuous, even if f is continuous.

Before calculating h, we want to show that

$$g : \mathbb{R} \longrightarrow [0,1[\,, \textit{ is one-to-one and onto.}$$

We write $\mathbb{R}$ as the union of disjoint intervals

$$\mathbb{R} = \{0\} \cup \left(\bigcup_{n=0}^{\infty} \left[-(n+1)^2, -n^2\right[\right) \cup \left(\bigcup_{n=0}^{\infty}]n, n+1] \right).$$

- *If $x = 0$, then $g(x) = 0$.*
- *If $x \in [-(n+1)^2, -n^2[$, $n = 0, 1, 2, \ldots$, then*

$$g(x) = \frac{\arctan(n) - \arctan(x)}{\pi}$$

This implies:

a) *g is one-to-one on $[-(n+1)^2, -n^2[$.*

b) *Since the* arctan *is increasing and continuous on $\mathbb{R}$ for $n = 0, 1, 2, \ldots$, the image of $[-(n+1)^2, -n^2[$ is the set*

$$\begin{aligned} A_n :&= g\left(\left[-(n+1)^2, -n^2\right[\right) \\ &= \left] \frac{\arctan(n) + \arctan(n^2)}{\pi}, \frac{\arctan(n) + \arctan((n+1)^2)}{\pi} \right]. \end{aligned}$$

- *If* $x \in \left]n, n+1\right]$, $n = 0, 1, 2, \ldots$, *then*

$$g(x) = \frac{\arctan(x) + \arctan((n+1)^2)}{\pi}.$$

This implies:

a) *g is one-to-one on* $\left]n, n+1\right]$.

b) *Since the* arctan *is increasing and continuous on* $\mathbb{R}$, *for* $n = 0, 1, 2, \ldots$ *the image of* $\left[n, n+1\right[$ *is the set*

$$\begin{aligned} B_n &:= g\left((n, n+1]\right) \\ &= \left] \frac{\arctan(n) + \arctan((n+1)^2)}{\pi}, \frac{\arctan(n+1) + \arctan((n+1)^2)}{\pi} \right]. \end{aligned}$$

Now, we define the following:

$$\begin{aligned} A &= \bigcup_{n=0}^{\infty} A_n \\ B &= \bigcup_{n=0}^{\infty} B_n, \\ I &= \{0\} \cup A \cup B. \end{aligned}$$

Then one can check that any two among the sets A_n, B_m *and* $\{0\}$, $n = 0, 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, *are disjoint, and*

$$I = \left[0, 1\right[.$$

Hence, g is one-to-one, and onto $[0,1[$.

To find $h : [0,1[\longrightarrow \mathbb{R}$ such that $f = h \circ g$, we recall that, if $t \in \mathrm{Im}(g)$ then $h(t) = f(g^{-1}(t))$,(i.e if $t = g(x)$ for some $x \in \mathbb{R}$, then $h(t) = f(x)$).

Now we can define our function $h : [0,1[\longrightarrow \mathbb{R}$.

- *If $t = 0$, then $t = g(0)$ and*

$$h(0) = f(0) = -1.$$

- *Let $t \in A_n$ for some $n = 0, 1, 2, \ldots$. Then there exists $x \in [-(n+1)^2, -n^2[$, such that $g(x) = t$, so*

$$t = \frac{\arctan(n) - \arctan(x)}{\pi}.$$

  This implies

$$x = \tan\left(\arctan n - \pi t\right).$$

  Hence

$$h(t) = f(x) = -1 + \sqrt{\tan\left(\pi t - \arctan n\right)}.$$

- *Let $t \in B_n$ for some $n = 0, 1, 2, \ldots$. Then there exists $x \in]n, n+1]$ such that $t = g(x)$. Hence*

$$h(t) = f(x) = n.$$

Now we can define the function $h : [0,1[\, \longrightarrow \mathbb{R}$, *as follows*

$$h(t) = \begin{cases} -1, & t = 0, \\ n, & t \in \left] \frac{\arctan(n)+\arctan((n+1)^2)}{\pi}, \frac{\arctan(n+1)+\arctan((n+1)^2)}{\pi} \right], \\ -1+\sqrt{\tan(\pi t - \arctan n)}, & t \in \left] \frac{\arctan(n)+\arctan(n^2)}{\pi}, \frac{\arctan(n)+\arctan((n+1)^2)}{\pi} \right]. \end{cases}$$

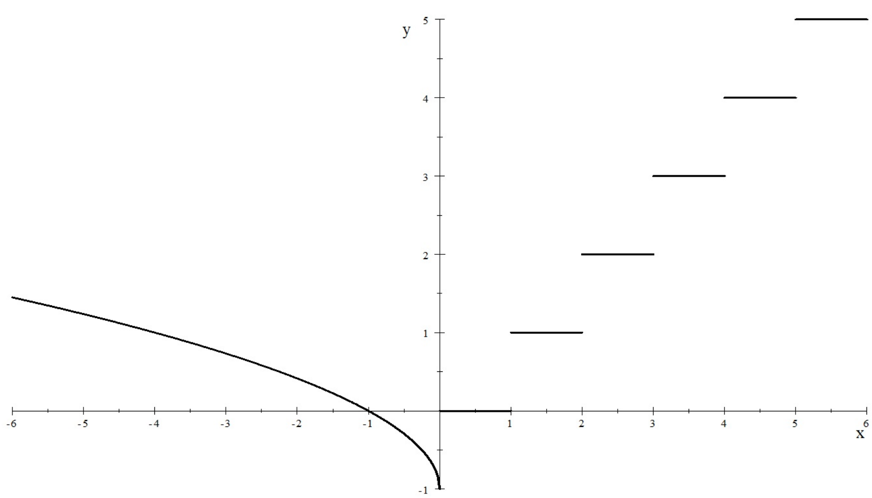

Figure 4.2: Graph of $y = f(x)$

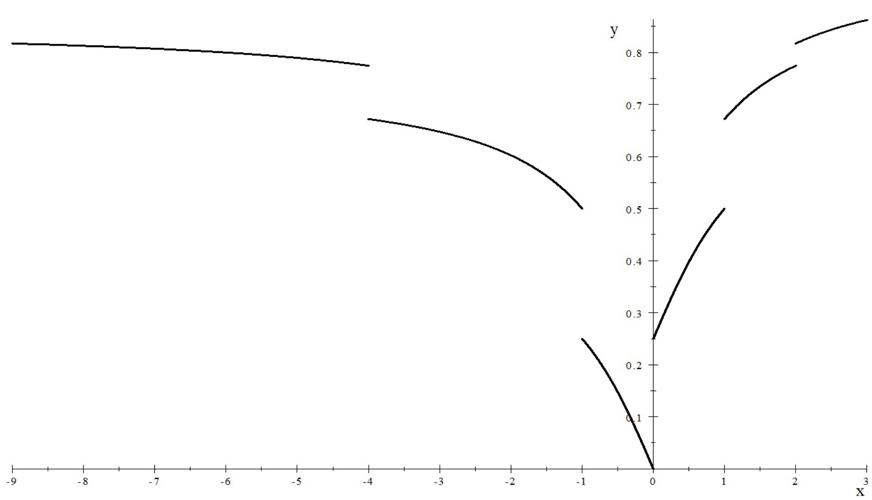

Figure 4.3: Graph of $y = g(x)$

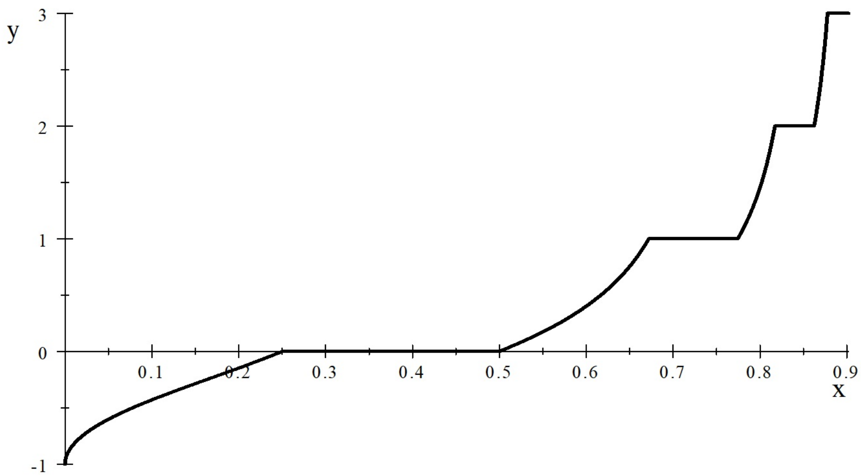

Figure 4.4: Graph of $y = h(x)$

4.4 Further Properties

A continuous quasiconvex function g such that every local minimum is global minimum, is necessarily semistrictly quasiconvex. This has been proved in [3, Th. 3.37] based on another result and on the separation theorem. We present here a simpler proof of this fact.

Proposition 4.8 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be neatly quasiconvex and continuous. Then f is semistrictly quasiconvex.*

Proof. Assume that f is neatly quasiconvex and continuous. Let $x, y \in \operatorname{dom} f$ be such that $f(x) < f(y)$. The set $S_f^<(y)$ is open and convex. Since y cannot be a local minimum, we have $y \in \overline{S_f^<(y)}$. As $x \in S_f^<(y)$, we deduce that $[x, y[\subseteq S_f^<(y)$, i.e., $f(z) < f(y)$ for all $z \in [x, y[$. Thus, f is semistrictly quasiconvex. ∎

Note that the quasiconvex function f of Example 4.1 that cannot be decomposed as $h \circ g$ with h nondecreasing and g semistrictly quasiconvex, is not continuous. A natural question arises: If f is quasiconvex and continuous, is there a decomposition $f = h \circ g$ such that h is nondecreasing, and g neatly quasiconvex and continuous, thus semistrictly quasiconvex? The answer is no, as shown by the following.

Example 4.6 *Consider the function f from Example 3.8.*

$$f(x,y) = \begin{cases} \max\{x,y\}, & x < 0 \text{ and } y < 0, \\ 0, & x \geq 0 \text{ and } y < 0, \\ y, & \text{elsewhere.} \end{cases}$$

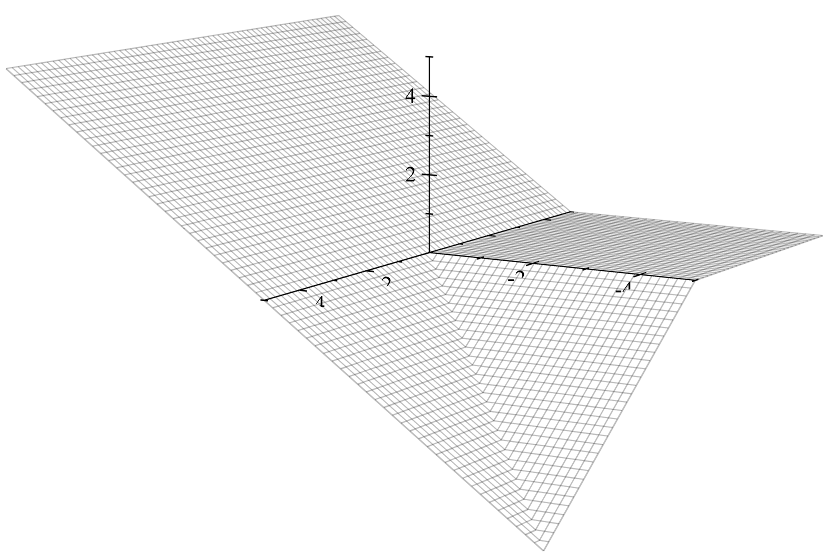

Figure 4.5: Graph of $f(x,y)$

It is quasiconvex and continuous.

Assume that we can write $f = h \circ g$ where h is nondecreasing and g is neatly quasiconvex and continuous. Then by Proposition 4.8, g is semistrictly quasiconvex. Let us show that g is constant on the set $A = (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_- \times \{0\})$.

For any two points $x, y \in A$, set

$$x_k = x - \frac{1}{k}\,(1,1)\,,$$

and,

$$y_k = y - \frac{1}{k}\,(1,1)\,.$$

Since at least one of the coordinates of the points x, y is 0, we have

$$f(x_k) = f(y_k) = -\frac{1}{k}.$$

Then

$$f(x_k) < f(y_{k+1}) < f(x_{k+2}),$$

from which we deduce

$$g(x_n) < g(y_{n+1}) < g(x_{n+2}).$$

By continuity, $g(x) = g(y)$.

Consider the points $a = (-1,-1)$, $b = (1,0)$ and $c = \left(0, -\frac{1}{2}\right)$. From $f(a) < f(b)$ we obtain $g(a) < g(b)$; by semistrict quasiconvexity, $g(c) < g(b)$. Now for every $t > 0$, $f(0,t) = t > 0 = f(b)$, so $g(0,t) > g(b)$. Taking the limit as $t \to 0$

we find $g(0,0) \geq g(b) > g(c) = g(0,0)$, *a contradiction.*

In Example 4.5 we can see that the function h in the composition $f = h \circ g$, is continuous, although the function f is discontinuous. We will show now, what are the conditions that imply the continuity of h. We first show the following.

Proposition 4.9 *Let* $W \subseteq \mathbb{R}$, *and let* $h : W \longrightarrow \mathbb{R}$ *be a non decreasing function. If* $\mathrm{Im}\,(h)$ *is an interval, then* h *is continuous on* W.

Proof. Assume for contradiction that h is discontinuous at a point $t_0 \in W$. Then there exists $\{t_k, k \in \mathbb{N}\} \subseteq W$, such that

$$t_k \longrightarrow t_0$$

and $h\,(t_k)$ does not converge to $h\,(t_0)$.

Without loss of generality we may assume that $t_k < t_0$ for all $k \in \mathbb{N}$. Then $h(t_k) \leq h(t_0)$, $k \in \mathbb{N}$. Set $c = \sup\,\{h\,(t_k)\}$, then $h\,(t_k) \longrightarrow c$.

Since h is discontinuous at t_0,

$$c < h(t_0). \tag{4.2}$$

Since $\mathrm{Im}\,(h)$ is an interval, then

$$]c, h\,(t_0)[\, \subseteq \mathrm{Im}\,(h)\,,$$

so there exist $d \in]c, h(t_0)[$ and $u \in \mathrm{dom} h$, such that

$$h(u) = d.$$

So, we have

$$h(t_k) < h(u) < h(t_0) \text{ for all } k,$$

this implies

$$t_k < u < t_0 \text{ for all } k.$$

But $t_k \longrightarrow t_0$, so $u = t_0$, which is a contradiction. ∎

Corollary 4.7 *Let the set $\Omega \subseteq \mathbb{R}^n$ be convex, and let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. If $\mathrm{Im}(f)$ is an interval in $\mathbb{R}$, then the function $h : \mathrm{Im}(g) \to \mathbb{R}$ that is given in Theorem 4.4 is continuous on $\mathrm{Im}(g)$.*

Proof. It is obvious that $\mathrm{Im}(h) = \mathrm{Im}(g)$. By applying the proposition we obtain that h is continuous. ∎

It should be noted that the result of the corollary does not depend on the choice of function α that was used in the proof of Theorem 4.4. It does not even depend on the particular construction of g that was used in the same theorem.

In the following example we show that h can be discontinuous if $\mathrm{Im}(f)$ is not an interval, even if f is quasiconvex and lower-semicontinuous.

Example 4.8 *Let* $f : [0,1] \longrightarrow \mathbb{R}$ *be given as*

$$f(x) = \begin{cases} x, & x \leq \frac{1}{2} \\ 5, & x > \frac{1}{2}. \end{cases}$$

Then f *is quasiconvex, and lower-semicontinuous.*

Now, take $\alpha : [0,1] \longrightarrow \mathbb{R}_{++}$ *defined by* $\alpha(x) = 1$ *for all* $x \in [0,1]$, *then*

$$S_f^a(x) = [0,x] \text{ and } g(x) = \int\limits_{S_f^a(x)} \alpha(t)\, dt = \int\limits_0^x \alpha(t)\, dt = x, \text{ for all } x \in [0,1].$$

This implies

$$h(t) = \begin{cases} t, & t \leq \frac{1}{2} \\ 5, & t > \frac{1}{2}. \end{cases}$$

Hence, $h : \mathrm{Im}(g) \longrightarrow \mathbb{R}$ *is not continuous at* $t = \frac{1}{2}$.

CHAPTER 5

APPROXIMATING QUASICONVEX FUNCTIONS WITH NEATLY QUASICONVEX ONES

In this chapter, our objective is to approximate any l.s.c. quasiconvex function by a better quasiconvex function, where every local minimum is a global minimum.

5.1 Preliminaries

Before we introduce our approximation function, we recall and prove some simple results, that we need in this chapter. Also, we will set the notation.

In this chapter we assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and convex.

Our first result is a trivial consequence of Proposition 4.4.

Proposition 5.1 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function. If $y \in S_f^a(x)$, then $f(y) \leq f(x)$.*

Proof. Obvious consequence of $y \in S_f^a(x) \subseteq S_f(x)$ and the definition of $S_f(x)$.

∎

Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Then for any $m = 0, 1, 2, \ldots, n$ we define:

- $\mathcal{C}_m = \left\{ S_f^a(x) : x \in \Omega \text{ and } \dim S_f^a(x) = m \right\}.$
- If $\mathcal{C}_m \neq \emptyset$ then

$$V_m = \operatorname{aff}(A) \text{ for any } A \in \mathcal{C}_m.$$

  It is clear from Proposition 4.7 that V_m is unique for each $m \in \{0, 1, 2, ..., n\}$.

- $W_m = \Omega \cap V_m$.
- For each $m \in \{1, 2, ..., n\}$, we take α_m on W_m, to be a continuous, positive function, with $\int_{W_m} \alpha_m(x) d\mu_m = 1$ (μ_m the Lebesgue measure on W_m).
- We define a function $M : \Omega \longrightarrow \mathbb{R}$ which satisfies a kind of monotonicity. For each $x \in \Omega$, if $\dim(S_f^a(x)) = m$, then we define

$$M(x) = \int_{S_f^a(x)} \alpha_m(\tau) d\mu_m + m. \tag{5.1}$$

In the next proposition, we will show what we mean by the monotonicity of the function M .

Proposition 5.2 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. If $S_f^a(x) \subsetneq S_f^a(y)$, then $M(x) < M(y)$.*

Proof. Since $S_f^a(x) \subsetneq S_f^a(y)$, we can say

$$\dim S_f^a(x) \leq \dim S_f^a(y).$$

Also, we have

$$\operatorname{ri}\left(S_f^a(y) \backslash S_f^a(x)\right) \neq \emptyset. \tag{5.2}$$

Furthermore

$$\operatorname{ri} S_f^a(y) \neq \emptyset. \tag{5.3}$$

If $\dim S_f^a(x) = m < p = \dim S_f^a(y)$, then by (5.2) (5.3) and the positivity of α_i for all $i \in \{1, 2, ..., n\}$, we can say

$$M(x) = \int\limits_{S_f^a(x)} \alpha_m(\tau) d\mu_m + m \leq p < p + \int\limits_{S_f^a(y)} \alpha_p(\tau) d\mu_p = M(y).$$

If $\dim S_f^a(x) = m = \dim S_f^a(y)$, then by (5.2) (5.3) and the positivity of α_i for all $i \in \{1, 2, ..., n\}$, we can say

$$M(x) = \int\limits_{S_f^a(x)} \alpha_m(\tau) d\mu_m + m < m + \int\limits_{S_f^a(y)} \alpha_m(\tau) d\mu_m = M(y).$$

In both cases, we have

$$M(x) < M(y).$$

∎

Proposition 5.2 will help us to construct an approximation $\{f_k\}$ for the function f, where f_k has no flat parts for all $k \in \mathbb{N}$.

Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Then for each $k \in \mathbb{N}$, we define a function $f_k : \Omega \longrightarrow \mathbb{R}$, where

$$f_k(x) = f(x) + \frac{M(x)}{k}, \text{ for all } x \in \Omega. \tag{5.4}$$

5.2 Properties and Convergence of f_k

In this section we will study the generalized convexity of the functions f_k, that were defined in (5.4). Also, we will prove the convergence of the sequence $\{f_k\}$.

In the next two propositions, we will present some relations between f_k, M, f, and S_f^a.

Proposition 5.3 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Let further $x, y \in \Omega$.*

a) *If $f(x) < f(y)$, then $M(x) < M(y)$, and $f_k(x) < f_k(y)$ for all $k \in \mathbb{N}$.*

b) *$f_k(x) \leq f_k(y)$ for any $k \in \mathbb{N}$ (for all $k \in \mathbb{N}$), if and only if $f(x) \leq f(y)$ and $M(x) \leq M(y)$.*

c) *$f_k(x) < f_k(y)$ for any $k \in \mathbb{N}$ (for all $k \in \mathbb{N}$), if and only if $f(x) \leq f(y)$ and $M(x) < M(y)$.*

d) *$f_k(x) = f_k(y)$ for any $k \in \mathbb{N}$ (for all $k \in \mathbb{N}$), if and only if $f(x) = f(y)$ and $M(x) = M(y)$.*

e) *$S_f^a(x) = S_{f_k}(x)$ for all $k \in \mathbb{N}$.*

Proof.

a) Assume that $f(x) < f(y)$, this implies

$$S_f^a(x) \subseteq S_f(x) \subseteq S_f^<(y) \subseteq S_f^a(y).$$

But $y \notin S_f^a(x)$, so

$$S_f^a(x) \subsetneq S_f^a(y).$$

Hence, from Proposition 5.2, we have

$$M(x) < M(y).$$

Then

$$f_k(x) < f_k(y), \text{ for all } k \in \mathbb{N}.$$

b) From Part (a), if $f(x) > f(y)$, then $f_k(x) > f_k(y)$ for all $k \in \mathbb{N}$, which contradicts the assumption that $f_k(x) \leq f_k(y)$.

Hence, $f(x) \leq f(y)$.

To prove that $M(x) \leq M(y)$, we consider two possible cases:

Case (1): $f(x) = f(y)$, then from the assumption $f_k(x) \leq f_k(y)$ we obtain

$$f(x) + \frac{M(x)}{k} \leq f(y) + \frac{M(y)}{k} \text{for all } k \in \mathbb{N}.$$

Thus, $M(x) \leq M(y)$.

Case (2): $f(x) < f(y)$. It follows from Part (a) that $M(x) < M(y)$.

Conversely, if $f(x) \leq f(y)$ and $M(x) \leq M(y)$, then it is obvious that

$$f_k(x) \leq f_k(y) \text{ for all } k \in \mathbb{N}.$$

c) If $f_k(x) < f_k(y)$ for any $k \in \mathbb{N}$, then from Part (b) we have

$$f(x) \leq f(y).$$

If $f(x) = f(y)$, then from $f_k(x) < f_k(y)$ we obtain

$$f(x) + \frac{M(x)}{k} < f(y) + \frac{M(y)}{k}$$

so $M(x) < M(y)$.

If $f(x) < f(y)$, then by Part (a), we have $M(x) < M(y)$.

Conversely, if $f(x) \leq f(y)$ and $M(x) < M(y)$, then from the definition of f_k we have

$$f_k(x) < f_k(y) \text{ for all } k \in \mathbb{N}.$$

d) If $f_k(x) = f_k(y)$, then

$$f_k(x) \leq f_k(y) \text{ and } f_k(x) \geq f_k(y).$$

Hence, from Part (b) we have

$$f(x) = f(y) \text{ and } M(x) = M(y). \tag{5.5}$$

The converse is obvious.

e) Assume that $z \in S_f^a(x)$. The following equivalences hold:

$$z \in S_f^a(x) \text{ if and only if}$$

$$f(z) \leq f(x) \text{ and } S_f^a(z) \subseteq S_f^a(x)\text{, if and only if}$$

$$f(z) \leq f(x) \text{ and } M(z) \leq M(z)\text{, if and only if (by Part (b))}$$

$$f_k(z) \leq f_k(x) \text{ for all } k \in \mathbb{N}\text{, if and only if}$$

$$z \in S_{f_k}(x) \text{ for all } k \in \mathbb{N}.$$

Hence,

$$z \in S_f^a(x) \text{ if and only if } z \in S_{f_k}(x),$$

so

$$S_f^a(x) = S_{f_k}(x) \text{ for all } k \in \mathbb{N}.$$

∎

Proposition 5.4 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Let further $x, y \in \Omega$. Then the following are equivalent*

a) $f_k(x) = f_k(y)$, for some $k \in \mathbb{N}$ (for all $k \in \mathbb{N}$).

b) $M(x) = M(y)$.

c) $S_f^a(x) = S_f^a(y)$.

Proof.

a) $\Longrightarrow$ **b)** If $f_k(x) = f_k(y)$ for some $k \in \mathbb{N}$, then from Part d of Proposition 5.3, we have

$$M(x) = M(y).$$

b) $\Longrightarrow$ **c)** If $M(x) = M(y)$, then from 5.1 and Proposition 4.3, we have

$$S_f^a(x) = S_f^a(y).$$

c) $\Longrightarrow$ **a)** If $S_f^a(x) = S_f^a(y)$, then from the definition of M, we have

$$M(x) = M(y). \tag{5.6}$$

Also, from Proposition 5.1 we have

$$f(x) \le f(y) \text{ and } f(y) \le f(x),$$

$$f(x) = f(y). \tag{5.7}$$

Hence, by (5.6) and (5.7), we have

$$f_k(x) = f_k(y) \text{ for all } k \in \mathbb{N}.$$

$\blacksquare$

Now, we will use the previous results to prove the quasiconvexity of f_k.

Theorem 5.1 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Then f_k as defined in (5.4) is quasiconvex, for all $k \in \mathbb{N}$.*

Proof. By Proposition 5.3(e) we have $S_{f_k}(x) = S_f^a(x)$, for all $k \in \mathbb{N}$. In view of Theorem 2.13, $S_{f_k}(x)$ is convex for all $x \in \text{dom}(f)$. Since $\text{dom}(f) = \text{dom}(f_k)$, Proposition 2.3 shows that f_k is quasiconvex for all $k \in \mathbb{N}$. $\blacksquare$

One of the nice and helpful properties of the function f_k, is that $\arg\min f_k = \arg\min f$ under some conditions, as we will see in the next proposition.

Proposition 5.5 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. If f_k is defined as in (5.4) , then*

$$\arg\min f = \arg\min f_k, \textit{ for all } k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$. If $x \in \arg\min f_k$, then by Part (b) of Proposition 5.3,

$$f(x) \leq f(y) \text{ for all } y \in \Omega.$$

Hence,

$$\arg\min f_k \subseteq \arg\min f.$$

Now, if $x \in \arg\min f$, then

$$S_f^a(x) \subseteq S_f^a(y) \text{ for all } y \in \Omega.$$

This implies

$$M(x) \leq M(y) \text{ for all } y \in \Omega.$$

From the last inequality, and our assumption that $x \in \arg\min f$, we have

$$f_k(x) \leq f_k(y) \text{ for all } y \in \Omega.$$

$$\text{i.e } \arg\min f \subseteq \arg\min f_k.$$

Hence,

$$\arg\min f = \arg\min f_k.$$

∎

In the next Theorem we will prove the main property of the function f_k, that f_k has no flat parts for each $k \in \mathbb{N}$, in other words; f_k is neatly quasiconvex for all $k \in \mathbb{N}$.

Theorem 5.2 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. If f_k is defined as in (5.4), then f_k is neatly quasiconvex for all $k \in \mathbb{N}$.*

Proof. Assume that $k \in \mathbb{N}$, and $x \in \Omega \backslash \arg\min f_k = \Omega \backslash \arg\min f$. We consider two cases:

Case (1): $x \in \bar{S}_f^<(x)$. Then for all $\varepsilon > 0$ there exists $y \in B(x, \varepsilon)$, such that $f(y) < f(x)$. By Part (a) of Proposition 5.3, we have $f_k(y) < f_k(x)$. Hence x is not a local minimum for f_k.

Case (2): $x \notin \bar{S}_f^<(x)$. Let x_0 be the projection of x on $\bar{S}_f^<(x)$. By the quasiconvexity of f, we have

$$f(z) \leq f(x), \text{ for all } z \in]x_0, x[.$$

For all $\varepsilon > 0$, we can find $z \in]x_0, x[\cap B(x, \varepsilon) \subseteq S_f^a(x)$. As $x \notin S_f^a(z)$, this implies

$$S_f^a(z) \subsetneq S_f^a(x).$$

Then

$$M(z) < M(x).$$

From Part (c) of Proposition 5.3, we have

$$f_k(z) < f_k(x).$$

Hence, in all cases, if $x \in \Omega \backslash \arg\min f_k$, then x is not a local minimum for f_k. So, f_k is neatly quasiconvex. ∎

In the following example, we show that the function f_k may not be strictly quasiconvex, even if f is quasiconvex and lower semi-continuous.

Example 5.3 *In $\mathbb{R}^2$, take $\Omega = \overline{B}(0, 10)$. and define $f : \Omega \longrightarrow \mathbb{R}$ by*

$$f(q,t) = \begin{cases} 0, & t = 0, \\ 5, & t \neq 0. \end{cases} \quad \textit{for all } x = (q,t) \in \Omega. \tag{5.8}$$

If $x \in [-1, 1] \times \{1\}$, then

$$S_f^a(x) = \overline{B}(0, 10) \cap ([-10, 10] \times [-1, 1]), \textit{ for all } x \in [-1, 1] \times \{1\}.$$

This implies

$$M(x) \textit{ is constant, for all } x \in [-1, 1] \times \{1\}. \tag{5.9}$$

Hence, from (5.8) and (5.9), we have

$$f_k(x) \textit{ is constant for all } x \in [-1, 1] \times \{1\}.$$

This implies that f_k is not strictly quasiconvex.

From the previous results we obtain the main result of this chapter.

Theorem 5.4 *Let $f : \Omega \longrightarrow \mathbb{R}$ be a quasiconvex function, such that $S_f(x)$ is closed for all $x \in \Omega$. Then there exists a sequence $\{f_k\}$ of neatly quasiconvex func-*

tions, such that f_k converges uniformly to f. Moreover $|f_k(x) - f(x)| \leq \frac{\dim(\Omega)+1}{k}$ for all $x \in \Omega$, and for all $k \in \mathbb{N}$.

Proof. Take the sequence $\{f_k\}$ such that

$$f_k(x) = f(x) + \frac{M(x)}{k} \text{ for all } x \in \Omega.$$

Then

$$|f_k(x) - f(x)| = \left|\frac{M(x)}{k}\right| \leq \frac{\dim\left(S_f^a(x)\right) + 1}{k} \leq \frac{\dim(\Omega) + 1}{k},\ k \in \mathbb{N}.$$

Hence, f_k converges uniformly to f. ∎

By comparing between Theorem 5.4 and [16, Thm 3.1], we can see that, in [16] they assume that the function f is continuous and bounded. In Theorem 5.4 we assume that $S_x(x)$ is closed, and no boundness condition assumed on f. On the other hand, in Theorem 5.4 we obtained a sequence of neatly quasiconvex functions, but in [16], the obtain a sequence of strictly quasiconvex functions, which is better than the neatly quasiconvex functions.

Example 5.5 *Let $f : [-5,5] \longrightarrow \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 1, & x \in [-5,-1], \\ \sqrt{-x}, & x \in]-1,0], \\ \sqrt{x}, & x \in]0,1[, \\ 1, & x \in [1,2], \\ x-1, & x \in]2,5]. \end{cases}$$

To find $M(x)$ we take the positive function $\alpha : [-5,5] \longrightarrow \mathbb{R}_{++}$, where $\alpha(x) = \frac{1}{10}$. Then $S_f^a(x)$, $M(x)$, and $f_k(x)$ are given as follows

x	$S_f^a(x)$	$M(x)$	$f_k(x)$
$x \in [-5,-2]$	$[x,2]$	$\frac{2-x}{10}+1$	$1+\frac{2-x}{10k}+\frac{1}{k}=1+\frac{12-x}{10k}$
$x \in (-2,-1)$	$[x,-x]$	$\frac{-2x}{10}+1$	$1+\frac{-2x}{10k}+\frac{1}{k}=1+\frac{10-2x}{10k}$
$x \in [-1,0)$	$[x,-x]$	$\frac{-2x}{10}+1$	$\sqrt{-x}+\frac{-2x}{10k}+\frac{1}{k}=\sqrt{-x}+\frac{10-2x}{10k}$
$x=0$	$\{0\}$	0	0
$x \in (0,1)$	$[-x,x]$	$\frac{2x}{10}+1$	$\sqrt{x}+\frac{2x}{10k}+\frac{1}{k}=\sqrt{x}+\frac{10+2x}{10k}$
$x \in [1,2]$	$[-x,x]$	$\frac{2x}{10}+1$	$1+\frac{2x}{10k}+\frac{1}{k}=1+\frac{10+2x}{10k}$
$x \in (2,5]$	$[-5,x]$	$\frac{x+5}{10}+1$	$x-1+\frac{x+5}{10k}+\frac{1}{k}=x-1+\frac{x+15}{10k}$

Table 5.1: $S_f^a(x)$ and $f_k(x)$

CHAPTER 6

CONCLUSION

In this work, we introduce the class of neatly quasiconvex functions, that is quasiconvex functions with no flat parts in their graph, or in other words, quasiconvex functions for which every local minimum is a global minimum.

The main result in this thesis is Theorem 4.4, which states that any quasiconvex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ with closed sublevel sets (i.e $S_f(x)$ is closed for all $x \in \mathbb{R}^n$), can be written as a composition of a neatly quasiconvex function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a nondecreasing function $h : \operatorname{Im} g \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$f = h \circ g.$$

Furthermore, the class of the adjusted sublevel sets of the function f equals the class of the sublevel sets of the function g, more precisely $S_g(x) = S_f^a(x)$ for all $x \in \mathbb{R}^n$. We have also shown that a neatly quasiconvex function $g : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is semistrictly quasiconvex, if it is continuous.

The function $h : \operatorname{Im} g \longrightarrow \mathbb{R} \cup \{+\infty\}$ in the composition $f = h \circ g$, was proved to be continuous, if the range of f is an interval in $\mathbb{R}$.

I addition to that, we have studied the properties of the adjusted sublevel set operator, especially the continuity properties. We have shown that this operator ($S_f^a : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$) is lower semicontinuous, if the function f is quasiconvex with closed sublevel sets $S_f(x)$ for all $x \in \mathbb{R}^n$. Also we proved the same property for the strict sublevel set operator, without assuming the quasiconvexity on the function f. As a consequence, we have proved that the normal cone operator to the adjusted sublevel set, and the normal cone to the strict sublevel sets, is cone upper semicontinuous.

Finally, we have approximated a quasiconvex function with closed sublevel sets, by a uniformly convergent sequence of neatly quasiconvex functions.

REFERENCES

[1] Aussel, D. and Hadjisavvas, N., *Adjusted Sublevel Sets, Normal Operator, and Quasi-convex Programming*, SIAM Journal on Optimization 16, 358–367 (2005).

[2] Aussel, D. and Pistek, M., *Limiting Normal Operator in Quasiconvex Analysis*, Set-Valued and Variational Analysis 23, 669-685 (2015).

[3] Avriel, M., Diewert, W.E., Schaible, S. and Zang, I.: *Generalized Concavity*, Classics in Applied Mathematics. SIAM, Philadelphia, PA. (2010).

[4] Borde, J. and Crouzeix, J.-P., *Continuity properties of the normal cone to the level sets of a quasiconvex function.* Journal of Optimization Theory and Applications 66, 415–429 (1990).

[5] Connell, C. and Rasmusen, E., *Concavifying the QuasiConcave*, Journal of Convex Analysis 24, 1239-1262 (2017).

[6] Crouzeix, J.-P., Eberhard, A. and Ralph, D., *A geometrical insight on pseudoconvexity and pseudomonotonicity.* Mathematical Programming 123, 61–83 (2010).

[7] Crouzeix, J.-P., Eberhard, A. and Ralph, D., *(Convex) Level Sets Integration*. Journal of Optimization Theory and Applications 171, 865–886 (2016).

[8] Fenchel, W., *Convex cones, sets and functions* (Notes from lectures in Princeton, 1951). Princeton University (1953).

[9] Guerraggio, A. and Molho, E., *The origins of quasi-concavity: a development between mathematics and economics*. Historia Mathematica 31, 62-75 (2004).

[10] Hadjisavvas, N., Komlosi, S. and Schaible, S., *Handbook of generalized convexity and generalized monotonicity*, Nonconvex Optimization and its Applications. Springer, New York. (2005).

[11] Hu, S. and Papageorgiou, N. S., *Handbook of Multivalued Analysis, Vol. I: Theory*, Mathematics and Its Applications (Springer). Springer, New York. (1997).

[12] Karamadian, S. and Schaible, S., *Seven kinds of monotone maps*, Journal of Optimization Theory and Applications 166, 37–46 (1990).

[13] Luc, D. T. and Penot, J.-P, *Convergence of asymptotic directions*. Transactions of the American Mathematical Society 353, 4095-4121 (2001).

[14] Mereau, P. and Paquet, J. G., *Second order conditions for pseudoconvex functions*, SIAM Journal on Applied Mathematics 27, 131–137 (1974).

[15] Mordukhovich, B. S. and Nam, M. N., *An Easy Path to Convex Analysis and Applications*. Synthesis Lectures on Mathematics and Statistics (2014).

[16] Lucchetti, R. and Milasi, M., *Approximating Quasiconvex Functions with Strictly Quasiconvex Ones in Banach Space*, Set-Valued and Variational Analysis 25, 591-602 (2017).

[17] Ponstein, J., *Seven Kinds of Convexity*. SIAM Review, 9, pp 115-119, (1967).

[18] Rockafellar, R. T., *Convex Analysis*, Princeton university press, Princeton, (1972).

[19] Von Neumann, J., *Zur Theorie der Gesellschaftsspriele*. Mathematische Annalen, 100, 295-320 (1928).

Vitae

- Name: Loai Shaalan
- Nationality: Palestinian
- Date of Birth: January 9, 1987
- Email: *loay303@gmail.com*
- Permenant Address: Hebron, Palestine
- Bachelors of Science in Mathematics, from Hebron University, 2009
- Masters of Science in Mathematics from Palestine Polytechnic University, 2013
- Doctor of Philosophy in Mathematics from King Fahd University of Petroleum and Minerals, 2018